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# A unified theory of the dynamics of closed-loop supply chains

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## Abstract

We investigate the dynamics of a closed-loop supply chain with first-order auto-regressive (AR(1)) demand and return processes. We assume these two processes are cross-correlated. The remanufacturing process is subject to a random triage yield. Remanufactured products are considered as-good-as-new and used to partially satisfy market demand; newly manufactured products make up the remainder. We derive the optimal linear policy in our closed-loop supply chain setting to minimise the manufacturer's inventory costs. We show that the *lead-time paradox* can emerge in many cases. In particular, the auto- and cross-correlation parameters and variances of the error terms in the demand and the returns, as well as the remanufacturing lead time, all influence the existence of the lead-time paradox. Finally, we propose managerial recommendations for manufacturers.

*Keywords:* Supply Chain Management, Closed-loop Supply Chain, Vector Auto-Regressive Process, Order-Up-To Policy, Random Yield

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## 1. Introduction

Collection and recycling systems for post-consumer products have been established in many countries. For example, many countries have achieved high collection and recycling rates for post-consumer polyethylene terephthalate (PET) bottles. (Welle, 2011), and various PET bottle-to-bottle recycling technologies have been developed (Coelho et al., 2011; Welle, 2011). For instance, in Japan, the collection and recycling rate of PET bottles reached 86.9% in 2015 (CPBR, 2015), and bottle-to-bottle mechanical recycling technology is used to manufacture as-good-as-new PET bottles from only reused resin (Suntory Group, 2013). This movement is mainly motivated by the sustainability ethic and public concern rather than an economic perspective (Welle, 2011). Understanding the dynamics of closed-loop supply chains (CLSC) can help improve their operational performance and economic viability. However, due to the natural complexity of CLSCs, their dynamical behaviour is not well understood (Akçalı and Çetinkaya, 2011).

It is known that in traditional forward supply chains, reducing the lead time often reduces the bullwhip effect and almost always reduces the variance of the net stock levels (Hosoda and Disney, 2006). However, some authors have noticed that in CLSCs increasing the remanufacturing lead time sometimes *decreases* the cost. This phenomena is called *the lead-time paradox* of CLSCs. This paradox was first reported by van der Laan et al. (1999) and was investigated further by Inderfurth and van der Laan (2001). Hosoda et al. (2015) also found a lead-time paradox in a CLSC, albeit in a different setting. This paradox may have a significant impact on the operational design of the reverse logistics network. The lead-time paradox might lead us to believe, counterintuitively, that importing remanufactured PET bottles from geographically remote countries (with long lead times) is more economic than sourcing them locally (with short lead times). The purpose of this paper is to analyse a CLSC setting that is general enough to resolve the question of the presence of the lead-time paradox.

The structure of our paper is as follows. In Section 2, we review literature related to our CLSC model. In Section 3 we develop our model and derive some useful properties in our setting. In Section 4, we present the results of a numerical analysis, confirming our theoretical contributions. Conclusions, managerial insights and potential future research directions are presented in Section 5.

## 2. Literature review and contribution

The bullwhip effect is a well-known phenomenon in traditional supply chains (Lee et al., 1997). Defined as the increase in the variability of the production compared to the variability of the demand, it is often measured as a ratio of the variances (Chen et al., 2000; Disney and Towill, 2003). Bullwhip is an important measure, as the induced variability increases both idling and overtime and creates excess capacity requirements. Furthermore, it can increase inventory requirements in upstream suppliers. Wang and Disney (2016) provided a recent review of the bullwhip literature, highlighting the open research questions in the field. Their study also noted the first-order auto-regressive AR(1) demand that is commonly assumed in the literature (Lee et al., 2000; Chen et al., 2000; Alwan et al., 2003; Zhang, 2004; Kim and Ryan, 2003; Hosoda and Disney, 2006, for example) as it is representative of many real demand patterns (Lee et al., 2000; Hosoda et al., 2008; Ali et al., 2017).

Recently, the vector auto-regressive (VAR) demand process has gained attention due to its ability to model multi-product situations. Originally, the VAR model was established to investigate multiple time series data sets (see Box and Tiao, 1977; Tiao and Box, 1981, for example). To the best of our knowledge, Kurata et al. (2007) was the first to use the VAR model to analyse a supply chain management problem. Kurata et al. (2007) investigated the impact of risk pooling and bundling in a supply chain consisting of a supplier and two manufacturers. Chaharsooghi and Sadeghi (2008) used the VAR demand process for two products in a two-level supply chain to investigate the bullwhip effect. This work was further extended by Sadeghi (2015), who investigated the bullwhip effect in a two-product, two-level supply chain. Here, the VAR demand was forecasted using the exponential smoothing method.

Ratanachote (2011) studied a VAR model of a distribution network with  $n$  warehouses and found that not only is there a square root law for inventory costs when the order-up-to (OUT) policy is used to generate replenishment orders, but there is also a square root law for capacity (bullwhip) costs. Boute et al. (2013) developed an uncorrelated noise VAR demand model to study a multi-product supply chain. General stability conditions were obtained. Raghunathan et al. (2017) considered an  $n$  product VAR(1) model with contemporaneous correlation in the forecast errors. They found that a *super bullwhip effect* exists under demand pooling. That is, pooling can

actually amplify rather than mitigate order variance.

Although the VAR model is gaining popularity in studies of forward supply chains, its application to a CLSC setting does not seem to have been considered before. Akçalı and Çetinkaya (2011) argued that correlation between demands and returns is a natural assumption, as some portion of the demand will eventually form the returns. However, they noted that simple demand and return processes are often adopted to avoid as much modelling complexity as possible. Hosoda et al. (2015) established a cross-correlated demand and return model in a CLSC setting, but both the demand and returns were independently and identically distributed (i.i.d.) random variables.

The existence of the lead-time paradox was reported in van der Laan et al. (1999), Inderfurth and van der Laan (2001) and Hosoda et al. (2015). Van der Laan et al. (1999) adopted a continuous review  $(s, Q)$  policy for the manufacturer and a push/pull policy for the remanufacturer to investigate the impact of the remanufacturer's policy and the lead times on system-wide cost. Their numerical analysis showed that system-wide cost decreases monotonically in the remanufacturing lead time. This counterintuitive finding occurred when the remanufacturing lead time was less than the manufacturing lead time. Using an  $(s, Q)$  continuous review policy for the manufacturer and the push policy for the remanufacturer, Inderfurth and van der Laan (2001) argued that cost is convex in the remanufacturing lead time. Inderfurth and van der Laan (2001) proposed that the remanufacturing lead time should be considered a decision variable. Based on a numerical analysis, they concluded that depending on the cost parameters, the optimal remanufacturing lead time should be equal to, or longer than, the manufacturing lead time.

Hosoda et al. (2015) studied the periodic review OUT policy with cross-correlated i.i.d. demand and return processes using the standard deviation of the net stock levels as an indicator of the inventory cost. They concluded that when the remanufacturing lead time is shorter than the manufacturing lead time, the lead-time paradox emerges in the inventory cost. The inventory cost decreases monotonically as the remanufacturing lead time increases up to the manufacturing lead time. Once these two lead times are equal, the remanufacturing lead time no longer affects the inventory cost.

There is also evidence that, contrary to the lead-time paradox, shorter remanufacturing lead times result in lower costs in CLSCs. Using a control theory approach, Zhou and Disney (2006) investigated a CLSC model and found that shorter remanufacturing lead time reduces net stock variance. Further, the greater the proportion of returns, the smoother the produc-

tion of new products. Based on a systematic literature review and some experimental analysis, Cannella et al. (2016) concluded that shorter remanufacturing lead times mitigate the bullwhip effect. Furthermore, both Zhou and Disney (2006) and Cannella et al. (2016) concluded that a larger return rate can reduce the bullwhip effect and the inventory variance.

We establish an optimal linear policy in our CLSC setting to minimise inventory costs. We model a proportional random yield in the triage process of the auto- and cross-correlated returns. Our modelling setting is general enough to capture instances when the lead-time paradox exists, supporting van der Laan et al. (1999), Inderfurth and van der Laan (2001), and Hosoda et al. (2015), and when the lead-time paradox does not exist, supporting Zhou and Disney (2006) and Cannella et al. (2016). Our theoretical contribution effectively integrates the two schools of thought on the lead-time paradox, thus representing a unified theory for CLSCs. We reveal that the lead-time paradox can exist in the bullwhip effect, the capacity cost and the inventory cost<sup>1</sup>.

### 3. A closed-loop supply chain model

Our CLSC model is a periodic review backlog system with constant lead times facing stochastic demand and return processes. Figure 1 is a schematic of our CLSC, which consists of a manufacturer and a remanufacturer. As all studies on the lead-time paradox assume a continuous review system (i.e. van der Laan et al., 1999; Inderfurth and van der Laan, 2001), except Hosoda et al. (2015), further study of periodic review policy models seems prudent to determine the extent of the lead-time paradox in this setting. This may be especially true as many supply chains operate on a discrete time basis; see Potter and Disney (2010) and Disney et al. (2013). Unlimited capacity is assumed in both the manufacturing and remanufacturing processes, enabling us to ensure mathematical tractability of our CLSC model. This assumption also reflects that capacity for PET bottle-to-bottle recycling is readily available in many countries (Welle, 2011). The triage process at the remanufacturer is subject to a random yield. The manufacturer holds a finished goods

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<sup>1</sup>Even if the lead-time paradox exists, factors outside our model (such as in-transit inventory) might be more significant. Therefore, lengthening the remanufacturing lead time to enjoy the lead-time paradox requires careful consideration, as there may be unintended consequences.

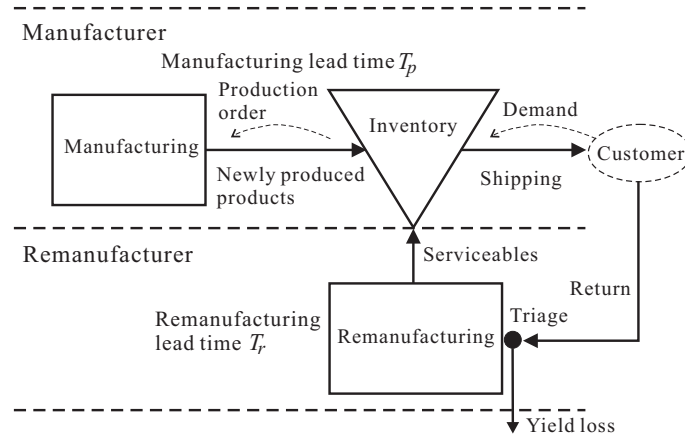


Figure 1: Schematic of the model

inventory and incurs an inventory holding or a backlog cost in each period. To minimise these linear convex inventory costs, the manufacturer exploits the OUT policy with a minimum mean square error (MMSE) forecast. This is known to be the optimal linear replenishment policy for our inventory cost function (Vassian, 1955; Hosoda and Disney, 2006; Hedenstierna and Disney, 2016).

The remanufacturer uses a push policy, as assumed by Inderfurth and van der Laan (2001). The push policy assumption for the remanufacturer fits well with the ethics of sustainability. Furthermore, the demand for recycled plastics is stable, despite the recent volatility in oil prices (PRE, 2016a), and it is reasonable to assume that a remanufacturer is motivated to use the push policy in order to quickly recover any costs associated with collecting and processing returns and avoid the costs of holding returns as inventory. The remanufactured but as-good-as new products are shipped to the manufacturer to partially meet the market demand. Any remaining demand is met by producing new products. The remanufacturing lead time,  $T_r$ , includes the transport time to the manufacturer, which might be influenced by the geographic size of the market.

We assume that the manufacturer has knowledge, via an information sharing strategy, of the return process and the yield rate and uses this knowledge to determine his production order quantity to minimise his inventory costs. This cooperative approach in the CLSC reflects the growing understanding that to enhance collection rates and recycling rates industry must work to-

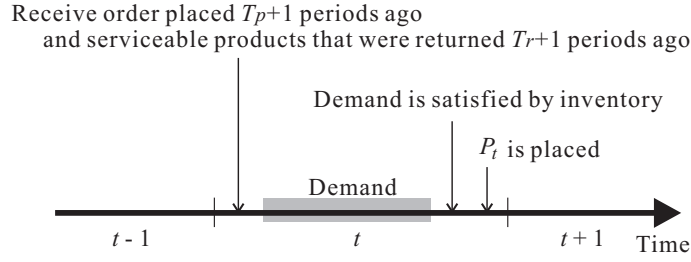


Figure 2: Sequence of events at the manufacturer

gether and share responsibility to create a sustainable society (Welle, 2011; CPBR, 2012; PRE, 2016b). The precise mechanism for information sharing we leave undefined; it could be enabled by ERP systems, shared files on the cloud or direct communication via telephone, fax or email.

### 3.1. Sequence of events

The sequence of events for the manufacturer is shown in Fig. 2. At the beginning of time period  $t$ , the manufacturer receives both the newly produced items from its production line and the serviceable items from the remanufacturer. The manufacturer then observes and satisfies the market demand. Unmet demand is backlogged. Finally, at the end of time period  $t$ , the manufacturer places a production order. This leads to the following net stock balance equation for the manufacturer,

$$NS_t = NS_{t-1} + \xi_{t-(T_r+1)} R_{t-(T_r+1)} + P_{t-(T_p+1)} - D_t. \quad (1)$$

Here,  $NS_t$  is the net stock at time  $t$ ,  $\xi_{t-(T_r+1)}$  is the yield rate realised at time  $t - (T_r + 1)$ ,  $T_r$  is the remanufacturing lead time,  $R_{t-(T_r+1)}$  is the returns received by the remanufacturer at time  $t - (T_r + 1)$ ,  $\xi_{t-(T_r+1)} R_{t-(T_r+1)}$  is the remanufactured products received by the manufacturer at  $t$ ,  $P_{t-(T_p+1)}$  is the production of new items completed after a production lead time of  $T_p$  and received by the manufacturer at  $t$  and  $D_t$  is the demand over time period  $t$ .

### 3.2. Demand and return

We adopt the vector auto-regressive process of the first order, VAR(1), to represent the case when both demand and returns are auto-correlated over time. The returns are cross-correlated with the demand but not vice versa;



this is a natural assumption, as products cannot be returned before demand has occurred. The demand and the return processes are defined as:

$$D_t = \mu_d + \phi_d(D_{t-1} - \mu_d) + \varepsilon_{d,t} \quad (2)$$

$$R_t = \mu_r + \phi_r(R_{t-1} - \mu_r) + \theta_r(D_{t-1} - \mu_d) + \varepsilon_{r,t}, \quad (3)$$

where  $\{\mu_d, \mu_r\}$  are the mean (average) demand and returns and  $\{\varepsilon_{d,t}, \varepsilon_{r,t}\}$  are i.i.d. random variables with zero means and standard deviations of  $\{\sigma_d, \sigma_r\}$ . We assume that  $\varepsilon_{d,t}$  and  $\varepsilon_{r,t}$  are independent of each other.  $\{\phi_d, \phi_r\}$  are the auto-correlation coefficients for the demand and returns, and  $\theta_r$  is the cross-correlation coefficient between demand in the previous period and the current returns. It is assumed that the manufacturer is aware of (3) in addition to (2) via an information sharing mechanism. Stability requires  $|\phi_d| < 1$  and  $|\phi_r| < 1$ . Interestingly, stability is independent of  $\theta_r$  (see Boute et al., 2013, for more information). While our analytical results hold for all stable systems, we assume that  $\phi_d \geq 0$  and  $\phi_r \geq 0$  when we conduct our numerical investigations. This reflects that most real demand processes have positive auto-regressive parameters (Lee et al., 2000; Hosoda et al., 2008; Ali et al., 2017).

Box et al. (2008) show the variance of the first order auto-regressive demand process is given by

$$V[D] = \frac{\sigma_d^2}{1 - \phi_d^2}.$$

A simple way to obtain this variance is shown in Appendix 1. The demand variance is finite when  $|\phi_d| < 1$ , infinite at  $|\phi_d| = 1$ , convex between these two points, and minimal at  $\phi_d = 0$ . The variance of the auto- and cross-correlated returns is given by

$$V[R] = \frac{\theta_r^2(1 + \phi_d\phi_r)V[D] + (1 - \phi_d\phi_r)\sigma_r^2}{(1 - \phi_d\phi_r)(1 - \phi_r^2)}. \quad (4)$$

Details of the process to obtain (4) are shown in Appendix 1. For a finite variance of the returns,  $|\phi_r| < 1$  and  $|\phi_d| < 1$  are required. When  $|\phi_d| = 1$  or  $|\phi_r| = 1$ , the return variance,  $V[R] \rightarrow \infty$ , as expected in an unstable system. Furthermore,  $V[R]$  is strictly increasing in  $\theta_r^2$ .

Our demand and return model allows  $R_t$  and  $D_{t-n}$  ( $n = 1, 2, 3, \dots$ ) to be correlated. The correlation between  $R_t$  and  $D_{t-n}$ ,  $\rho_n$  is given by

$$\rho_n = \frac{COV_n}{\sqrt{V[D]}\sqrt{V[R]}},$$

where  $COV_0 = \phi_d \theta_r V[D]/(1 - \phi_d \phi_r)$ ,  $COV_1 = \phi_r COV_0 + \theta_r V[D]$ , and

$$COV_{n \geq 2} = \begin{cases} \phi_r^n COV_0 + \theta_r \frac{\phi_d^n - \phi_r^n}{\phi_d - \phi_r} V[D], & \phi_d \neq \phi_r \\ \phi_r^n COV_0 + n \theta_r \phi_d^{n-1} V[D], & \phi_d = \phi_r \neq 0 \\ \theta_r \phi_r^{n-1} V[D], & \phi_d = 0 \wedge \phi_r \neq 0 \\ \theta_r \phi_d^{n-1} V[D], & \phi_d \neq 0 \wedge \phi_r = 0 \\ 0, & \phi_d = \phi_r = 0. \end{cases}$$

Details of the process to obtain these covariances are shown in Appendix 1.

When the cross-correlation coefficient  $\theta_r = 0$ , there is no correlation between  $R_t$  and  $D_{t-n}$  (i.e.  $\rho_n = 0$ ). If  $D_{t-n}$  is large (small) and is frequently followed by large (small) returns,  $R_t$ , the value of  $\rho_n$  is likely to be positive. If large (small) demand is frequently followed by small (large) returns, the value of  $\rho_n$  is likely to be negative. Negative correlation may occur when, for example, the total available logistic capacity is limited and this limited capacity is used to deliver finished products and collect returns. In such cases, high demand requires a larger proportion of the available logistics capacity, and because the capacity is limited, the capacity available for collecting returns is reduced. Besides the dynamic information sharing, we also assume that both parties have the capability to identify and share static information—the lead times and the underlying VAR(1) demand and return process parameters—and that the structure of the demand and return processes are unchanging over time.

### 3.3. Random yields in the triage process

A random yield in the remanufacturing process is a natural assumption, as the returns may exhibit large variation in quality. We use the stochastically proportional yield model (Henig and Gerchak, 1990) to represent the random yield at the remanufacturer. The yield at time period  $t$ ,  $\xi_t$ , is an i.i.d. stochastic process, and the yield loss is proportional to the return quantity (i.e.  $(1 - \xi_t)R_t$ ), as in Hosoda et al. (2015). This proportional model is appropriate when the return is subject to material variations (Yano and Lee, 1995). No correlation is assumed between  $\xi_t$  and  $R_t$ , and no specific distribution for  $\xi_t$  is assumed. The uniform, triangular or beta distribution (amongst others) could be used to represent  $\xi_t$ . In what follows,  $\Xi[\cdot]$  is used to represent the yield from the triage process. For example,  $\Xi[R_t]$  represents the value of  $\xi_t R_t$ . The variance of the production of remanufactured items,  $V[\Xi[R]]$ , when  $R_t$

follows a VAR(1) process is

$$V[\Xi[R]] = V[\xi]\mu_r^2 + (V[\xi] + \bar{\xi}^2)V[R], \quad (5)$$

where  $V[\xi]$  is the variance of  $\xi_t$  and  $\bar{\xi}$  is the mean of  $\xi_t$ . The process to obtain  $V[\Xi[R]]$  is presented in Appendix 2. Note that  $V[\Xi[R]]$  includes the mean of the yield,  $\bar{\xi}$ , and the mean of the returns,  $\mu_r$ , which indicates that our model is non-linear and therefore cannot be analysed using the traditional control theory approach (Dejonckheere et al., 2003). Note that in the special case that the yield rate is constant (i.e.  $V[\xi] = 0$ ), the system remains linear (hence its popularity in previous studies),  $\mu_r$  disappears from  $V[\Xi[R]]$  and only  $\bar{\xi}$  and  $V[R]$  have an impact upon  $V[\Xi[R]]$ .

#### 3.4. Derivation of CLSC order-up-to policy

Hosoda et al. (2015) showed that regardless of the type of ordering policy present, the following relationship always holds in our CLSC setting:

$$\begin{aligned} NS_{t+T_p+1} &= NS_t + (P_t + \sum_{i=1}^{T_p} P_{t-i}) \\ &\quad - (D_{t+T_p+1} + \sum_{i=1}^{T_p} D_{t+i}) + PIR_t + FPIR_t. \end{aligned} \quad (6)$$

Here, the sum of the pipeline inventory of returns ( $PIR_t$ ) and the future pipeline inventory of returns ( $FPIR_t$ ),  $PIR_t + FPIR_t$ , represents the current total quantity of on-order remanufactured products at time period  $t$ .  $PIR_t$  and  $FPIR_t$  are defined as,

$$PIR_t = \begin{cases} \sum_{i=T_r-T_p}^{T_r} \Xi[R_{t-i}], & T_r \geq T_p \\ \sum_{i=0}^{T_r} \Xi[R_{t-i}], & T_r < T_p \end{cases}$$

and

$$FPIR_t = \begin{cases} 0, & T_r \geq T_p \\ \sum_{i=1}^{T_p-T_r} \Xi[R_{t+i}], & T_r < T_p. \end{cases}$$

It is assumed that the value of  $PIR_t$  is known by the manufacturer at time  $t$ , as its value is already realised and observed by the remanufacturer and the necessary information is shared with the manufacturer. However, when  $T_r < T_p$ , the value of  $FPIR_t$  is unknown at time  $t$ ; it will be revealed sometime in the future. In this case, the manufacturer must estimate its value. Equation 6

shows that the variability of the net stock level at time period  $t + T_p + 1$  originates from the uncertain future demands ( $\sum_{i=1}^{T_p+1} D_{t+i}$ ) and when  $T_r < T_p$ ,  $FPIR_t (= \sum_{i=1}^{T_p-T_r} \Xi[R_{t+i}])$ . This fact suggests that if  $T_p$  is constant, then the uncertainty of  $NS_{t+T_p+1}$  mainly comes from  $FPIR_t$  and can be reduced by lengthening  $T_r$ , as  $FPIR_t = \sum_{i=1}^{T_p-T_r} \Xi[R_{t+i}]$ .

We assume the following inventory cost function is relevant for the manufacturer:

$$J = hE[(NS_t)^+] + gE[(-NS_t)^+],$$

where  $h$  is the per period unit inventory holding cost and  $g$  is the per period unit backlog cost<sup>2</sup>. If it is reasonable to assume that  $NS_t$  follows the normal distribution<sup>3</sup>, to minimise  $J$ , the expected value of  $NS_{t+T_p+1}$  in (6) should be set to:

$$E[NS_{t+T_p+1}] = \sqrt{V[NS]} \Phi^{-1}\left[\frac{g}{g+h}\right] := TNS,$$

(Zipkin, 2000), where  $V[NS]$  is the variance of the net stock levels over an infinite time horizon defined as  $E[(NS_t - E[NS_t])^2]$ , and  $\Phi^{-1}[\cdot]$  is the inverse of the standard normal cumulative distribution function.  $TNS$  represents the target net stock and if used when setting production targets and inventory levels were normally distributed, the following expression would give the per period expected inventory and holding cost at the manufacturer (Zipkin, 2000):

$$J = \sqrt{V[NS]}(g+h)\varphi\left[\Phi^{-1}\left[\frac{g}{g+h}\right]\right], \quad (7)$$

where  $\varphi[\cdot]$  is the standard normal density function. Equation 7 shows that with optimal safety stocks, the inventory costs are a linear function of the standard deviation of the net stock levels. Therefore, understanding the standard deviation of the net stock suffices to understand the inventory costs.

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<sup>2</sup>We do not use  $b$  for the backlog cost to avoid confusion when we discuss the case of a triangular distribution for the triage yield in Section 4.

<sup>3</sup>The appropriateness of the normality assumption on the net stock levels is discussed in Appendix 3.

At time period  $t$ , (6) can be rewritten as:

$$\begin{aligned} TNS = & NS_t + (P_t + \sum_{i=1}^{T_p} P_{t-i}) \\ & - \left( E[D_{t+T_p+1}] + E\left[\sum_{i=1}^{T_p} D_{t+i}\right] \right) + PIR_t + E[FPIR_t]. \end{aligned} \quad (8)$$

Rearranging (8) reveals an optimal replenishment OUT policy for minimising the inventory cost in our CLSC:

$$\begin{aligned} P_t = & E[D_{t+T_p+1}] + \underbrace{TNS - NS_t}_{\text{Inventory feedback}} \\ & + \underbrace{E\left[\sum_{i=1}^{T_p} D_{t+i}\right] - \left(\sum_{i=1}^{T_p} P_{t-i} + PIR_t + E[FPIR_t]\right)}_{\text{WIP feedback}}. \end{aligned} \quad (9)$$

Note that (9) is an OUT policy. Another formulation of this OUT policy can be obtained as follows. Using the net stock balance equation (1),  $NS_t + \sum_{i=0}^{T_p} P_{t-i}$  can be rewritten as:

$$\begin{aligned} NS_t + \sum_{i=0}^{T_p} P_{t-i} &= \underbrace{NS_{t-1} + \Xi[R_{t-(T_r+1)}] + P_{t-(T_p+1)} - D_t}_{=NS_t} + \sum_{i=0}^{T_p} P_{t-i} \\ &= \underbrace{P_t + \sum_{i=0}^{T_p} P_{t-1-i}}_{=P_{t-(T_p+1)} + \sum_{i=0}^{T_p} P_{t-i}} + NS_{t-1} + \Xi[R_{t-(T_r+1)}] - D_t. \end{aligned} \quad (10)$$

Rearranging (10) results in:

$$\begin{aligned} P_t = & D_t - \Xi[R_{t-(T_r+1)}] \\ & + \left( NS_t + \sum_{i=0}^{T_p} P_{t-i} \right) - \left( NS_{t-1} + \sum_{i=0}^{T_p} P_{t-1-i} \right). \end{aligned} \quad (11)$$

To eliminate  $\{NS_t, NS_{t-1}\}$  and  $\{\sum_{i=0}^{T_p} P_{t-i}, \sum_{i=0}^{T_p} P_{t-1-i}\}$  on the right-hand side of (11), we again use (6), which results in:

$$NS_t + \sum_{i=0}^{T_p} P_{t-i} = \sum_{i=1}^{T_p+1} D_{t+i} - PIR_t - FPIR_t + NS_{t+T_p+1}. \quad (12)$$

Using expected values of the future variables in (12) and substituting these into (11) yields:

$$\begin{aligned} P_t = & D_t - \Xi[R_{t-(T_r+1)}] + \left( E \left[ \sum_{i=1}^{T_p+1} D_{t+i} \right] - PIR_t - E[FPIR_t] \right) \\ & - \left( E \left[ \sum_{i=1}^{T_p+1} D_{t-1+i} \right] - PIR_{t-1} - E[FPIR_{t-1}] \right), \end{aligned} \quad (13)$$

which is yet another form of OUT policy for our CLSC. This form is particularly useful because the net stock level is not required to calculate the order quantity, thus simplifying our analysis.

Let  $s$  be the slack capacity for the production. Let us assume that if the production order is greater than a regular capacity level of  $\mu_d - \bar{\xi}\mu_r + s$ , the excess production requirements are met by working overtime at a unit cost of  $w$ . If the production requirements do not fill the regular capacity level,  $\mu_d - \bar{\xi}\mu_r + s$ , a unit opportunity loss of  $u$  is incurred. Therefore, the per period capacity cost is given by:

$$C = uE[(\mu_d - \bar{\xi}\mu_r + s) - P_t]^+ + wE[(P_t - (\mu_d - \bar{\xi}\mu_r + s))^+].$$

Under this cost regime,  $s$  is a decision variable to be optimised (in a similar manner to the newsvendor model). Disney et al. (2012) showed that when production orders are normally distributed, the optimal slack capacity (above or below the mean demand,  $\mu_d$ , minus the mean serviceable returns,  $\bar{\xi}\mu_r$ ), is given by:

$$s^* = \sqrt{V[P]} \Phi^{-1} \left[ \frac{w}{u+w} \right],$$

where  $V[P]$  is the variance of the production order over an infinite time horizon. When a capacity of  $\mu_d - \bar{\xi}\mu_r + s^*$  and normally distributed production orders are present then the expected per period capacity cost is:

$$C = \sqrt{V[P]}(u+w)\varphi \left[ \Phi^{-1} \left[ \frac{w}{u+w} \right] \right]. \quad (14)$$

Equation 14 shows that the expected capacity cost is linear in the standard deviation of production orders. This demonstrates that understanding of the standard deviation of the orders is sufficient to understand capacity cost behaviour. Appendix 3 investigates the appropriateness of assuming that the production orders are normally distributed.

### 3.5. Order-up-to policy in a CLSC with minimum mean square error forecasts of demand and returns

As shown by Box et al. (2008), conditional expectation provides an MMSE forecast for an AR(1) process. Furthermore, the MMSE forecast minimises the variance of the net stock levels under the OUT policy (Vassian, 1955; Hosoda and Disney, 2006). The MMSE forecast of the AR(1) process over the lead time plus review period made at time  $t$  is well known (Lee et al., 2000; Hosoda and Disney, 2006):

$$E \left[ \sum_{n=1}^{T_p+1} D_{t+n} \right] = (T_p + 1)\mu_d + \phi_d \frac{1 - \phi_d^{T_p+1}}{1 - \phi_d} (D_t - \mu_d).$$

The MMSE forecast of the  $n$  ( $= 1, 2, 3, \dots$ ) period-ahead returns made at time  $t$  is

$$E[R_{t+n}] = \begin{cases} \mu_r + \phi_r^n (R_t - \mu_r) + \theta_r \frac{\phi_d^n - \phi_r^n}{\phi_d - \phi_r} (D_t - \mu_d), & \phi_d \neq \phi_r \\ \mu_r + \phi_r^n (R_t - \mu_r) + n\theta_r \phi_d^{n-1} (D_t - \mu_d), & \phi_d = \phi_r \neq 0 \\ \mu_r + \theta_r (D_t - \mu_d), & \phi_d = \phi_r = 0 \wedge n = 1 \\ \mu_r, & \phi_d = \phi_r = 0 \wedge n \geq 2, \end{cases}$$

which enables us to obtain the following expressions for the MMSE forecast of  $R_t$  over the lead time ( $T_p$ ) and the review period (+1):

$$E \left[ \sum_{n=1}^{T_p+1} R_{t+n} \right] = (T_p + 1)\mu_r + \frac{\phi_r(\phi_r^{T_p+1} - 1)}{\phi_r - 1} (R_t - \mu_r) + X(D_t - \mu_d),$$

where

$$X = \begin{cases} \frac{\theta_r(\phi_d^{T_p+2}(\phi_r-1) - \phi_r^{T_p+2}(\phi_d-1) + \phi_d - \phi_r)}{(\phi_d-1)(\phi_d-\phi_r)(\phi_r-1)}, & \phi_d \neq \phi_r \\ \frac{\theta_r(1 + \phi_d^{T_p+1}(\phi_d + T_p(\phi_d-1) - 2))}{(\phi_d-1)^2}, & \phi_d = \phi_r. \end{cases}$$

The MMSE forecast of  $FPIR_t$ ,  $E[FPIR_t]$  is

$$\begin{aligned} E[FPIR_t] &= E \left[ \sum_{i=1}^{T_p-T_r} \Xi[R_{t+i}] \right] = E \left[ \sum_{i=1}^{T_p-T_r} \xi_{t+i} R_{t+i} \right] \\ &= \bar{\xi} \left( (T_p - T_r)\mu_r + \sum_{i=1}^{T_p-T_r} \phi_r^i (R_t - \mu_r) \right. \\ &\quad \left. + \theta_r \sum_{i=0}^{T_p-T_r} \sum_{j=0}^{i-1} \phi_d^{i-j-1} \phi_r^j (D_t - \mu_d) \right) \\ &= \bar{\xi} \left( (T_p - T_r)\mu_r + \phi_r \frac{(\phi_r^{T_p-T_r} - 1)}{\phi_r - 1} (R_t - \mu_r) + \Lambda_\theta (D_t - \mu_d) \right), \end{aligned}$$

where

$$\Lambda_\theta = \begin{cases} \frac{\theta_r(\phi_d^{T_p-T_r+1}(\phi_r-1) - \phi_r^{T_p-T_r+1}(\phi_d-1) + \phi_d - \phi_r)}{(\phi_d-1)(\phi_d-\phi_r)(\phi_r-1)}, & \phi_d \neq \phi_r \\ \frac{\theta_r(1 + \phi_d^{T_p-T_r}((T_p-T_r)(\phi_d-1)-1))}{(\phi_d-1)^2}, & \phi_d = \phi_r. \end{cases}$$

Substituting these conditional expectations into (13) yields the OUT policy with the MMSE forecast for our CLSC. The ordering policy when  $T_r \geq T_p$  can be simplified to:

$$P_t = D_t + \phi_d \frac{1 - \phi_d^{T_p+1}}{1 - \phi_d} (D_t - D_{t-1}) - \Xi[R_{t-(T_r-T_p)}], \quad (15)$$

as all  $FPIR$ s are null in this case, and  $\Xi[R_{t-(T_r+1)}] + PIR_t - PIR_{t-1}$  is equal to  $\Xi[R_{t-(T_r-T_p)}]$ . Equation 15 shows that the manufacturer needs knowledge of  $\Xi[R_{t-(T_r-T_p)}]$  from the remanufacturer to determine the value of  $P_t$ .

The ordering policy for the case of  $T_p > T_r$  is:

$$P_t = D_t + \phi_d \frac{1 - \phi_d^{T_p+1}}{1 - \phi_d} (D_t - D_{t-1}) - \Xi[R_t] - \bar{\xi} \left( \phi_r \frac{\phi_r^{T_p-T_r} - 1}{\phi_r - 1} (R_t - R_{t-1}) + \Lambda_\theta (D_t - D_{t-1}) \right). \quad (16)$$

To obtain (16), we used  $\Xi[R_{t-(T_r+1)}] + PIR_t - PIR_{t-1} = \Xi[R_t]$ . In this case, the manufacturer needs additional dynamic information from the remanufacturer,  $\Xi[R_t]$  and  $R_t$ , in addition to the static information,  $\bar{\xi}$ ,  $\phi_r$  and  $\theta_r$ , to optimally determine his order quantity.

### 3.6. Variance of order rates

In order to determine if a bullwhip (or capacity cost) lead-time paradox exists, we need to inspect the production order variance. By substituting (2) into (15), when  $T_r \geq T_p$  the order can be written as:

$$P_t = D_t - \Xi[R_{t-(T_r-T_p)}] + \Lambda_d(\phi_d - 1)(D_{t-1} - \mu_d) + \Lambda_d \varepsilon_{d,t},$$

where  $\Lambda_d = \phi_d(1 - \phi_d^{T_p+1})/(1 - \phi_d)$ . Appendix 4 shows that the variance of the order rate when  $T_r \geq T_p$  is:

$$\begin{aligned} V[P] &= E[(P_t - E[P_t])^2] \\ &= V[D] + V[\Xi[R]] + 2\Lambda_d(1 - \phi_d^{T_p+2})V[D] - 2\bar{\xi}\phi_d^{T_p+1}COV_0, \end{aligned} \quad (17)$$



where  $COV_0$  is the covariance between  $D_t$  and  $R_t$ , as shown in (27) in Appendix 1. The variance of order rates in a traditional forward supply chain under AR(1) demand when an OUT policy and MMSE forecasting scheme is present is:

$$V[D] + 2\Lambda_d(1 - \phi_d^{T_p+2})V[D]; \quad (18)$$

see Hosoda and Disney (2006). Therefore, the order variance expression of our CLSC, (17), can be interpreted as the sum of the order rate's variance of the traditional forward supply chain, (18), and the variance of the serviceable returns ( $V[\Xi[R]]$ ) minus a function of the covariance between the demand and the returns ( $2\bar{\xi}\phi_d^{T_r+1}COV_0$ ).

**Property 1.** *When  $T_r \geq T_p$ ,  $\phi_d > 0$  and  $COV_0 > 0$ ,  $V[P]$  is increasing in both  $T_r$  and  $T_p$ .*

Property 1 reveals that when  $T_r \geq T_p$ ,  $\phi_d > 0$  and  $COV_0 > 0$ , as in a traditional forward supply chain, the order variance (and hence bullwhip and capacity costs) is increasing in both the lead times.

**Property 2.** *When  $T_r \geq T_p$  and  $\phi_d = 0$ ,  $V[P]$  is equal to  $V[D] + V[\Xi[R]]$  and is independent of  $\{T_r, T_p\}$ .*

Property 2 suggests that when  $T_r \geq T_p$ , because  $V[\Xi[R]] > 0$ , the bullwhip effect is always present (i.e.  $V[P] > V[D]$ ), even though the demand is a white noise process (i.e.  $\phi_d = 0$ ). A similar finding was shown in Hosoda et al. (2015) and is interesting because traditional supply chains with i.i.d. demand and MMSE forecasts exhibit a bullwhip ratio of unity. This suggests that returns are likely to introduce variability in the production volume of new products.

When  $T_r < T_p$ ,  $P_t$ , (16) can be rewritten as:

$$\begin{aligned} P_t = & D_t - \Xi[R_t] + ((\Lambda_d - \bar{\xi}\Lambda_\theta)(\phi_d - 1) - \bar{\xi}\Lambda_r\theta_r)(D_{t-1} - \mu_d) \\ & + \bar{\xi}\Lambda_r(1 - \phi_r)(R_{t-1} - \mu_r) + (\Lambda_d - \bar{\xi}\Lambda_\theta)\varepsilon_{d,t} - \bar{\xi}\Lambda_r\varepsilon_{r,t}, \end{aligned}$$

where  $\Lambda_r = \phi_r(1 - \phi_r^{T_p-T_r})/(1 - \phi_r)$ . The variance of  $P_t$  when  $T_r < T_p$  then

becomes:

$$\begin{aligned}
V[P] &= E[(P_t - E[P_t])^2] \\
&= E \left[ ((D_t - \mu_d) - (\Xi[R_t] - \bar{\xi}\mu_r) \right. \\
&\quad \left. + ((\Lambda_d - \bar{\xi}\Lambda_\theta)(\phi_d - 1) - \bar{\xi}\Lambda_r\theta_r)(D_{t-1} - \mu_d) \right. \\
&\quad \left. + \bar{\xi}\Lambda_r(1 - \phi_r)(R_{t-1} - \mu_r) + (\Lambda_d - \bar{\xi}\Lambda_\theta)\varepsilon_{d,t} - \bar{\xi}\Lambda_r\varepsilon_{r,t} \right)^2 \Big] \\
&= V[D] + V[\Xi[R]] + ((\Lambda_d - \bar{\xi}\Lambda_\theta)(\phi_d - 1) - \bar{\xi}\Lambda_r\theta_r)^2 V[D] \\
&\quad + \bar{\xi}^2 \Lambda_r^2 (1 - \phi_r)^2 V[R] + (\Lambda_d - \bar{\xi}\Lambda_\theta)^2 \sigma_d^2 + \bar{\xi}^2 \Lambda_r^2 \sigma_r^2 \\
&\quad - 2\bar{\xi} COV_0 + 2((\Lambda_d - \bar{\xi}\Lambda_\theta)(\phi_d - 1) - \bar{\xi}\Lambda_r\theta_r) \phi_d V[D] \\
&\quad + 2\phi_d \bar{\xi} \Lambda_r (1 - \phi_d) COV_0 + 2(\Lambda_d - \bar{\xi}\Lambda_\theta) \sigma_d^2 \\
&\quad - 2((\Lambda_d - \bar{\xi}\Lambda_\theta)(\phi_d - 1) - \bar{\xi}\Lambda_r\theta_r) \bar{\xi} COV_1 \\
&\quad - 2\bar{\xi}^2 \Lambda_r (1 - \phi_r) (\phi_r V[R] + \theta_r COV_0) \\
&\quad + 2((\Lambda_d - \bar{\xi}\Lambda_\theta)(\phi_d - 1) - \bar{\xi}\Lambda_r\theta_r) \bar{\xi} \Lambda_r (1 - \phi_r) COV_0 \\
&\quad + 2\bar{\xi}^2 \Lambda_r \sigma_r^2. \tag{19}
\end{aligned}$$

The relationships (24) and (28) derived in Appendix 1 were used to obtain the last expression. Equations (17) and (19) yield the following property.

**Property 3.**  $V[P]$  is increasing in  $\mu_r$ , irrespective of the values of  $\{T_p, T_r\}$ .

As discussed in Section 3.3,  $V[\Xi[R]]$ , (5), includes the mean of the returns,  $\mu_r$ . As shown in (17) and (19),  $V[P]$  includes  $V[\Xi[R]]$ . Therefore it is obvious that  $V[P]$  is increasing in  $\mu_r$ , suggesting that increasing the mean returns increases bullwhip and the capacity cost. While  $\bar{\xi}$  is contained in  $V[\Xi[R]]$ , unlike with  $\mu_r$ , we cannot infer that  $V[P]$  is increasing in  $\bar{\xi}$  because  $\bar{\xi}$  is also contained in  $V[\xi]$ . Indeed, as  $0 \leq \xi_t \leq 1$ ,  $V[\xi]$  and  $V[\Xi[R]]$  is often decreasing in  $\bar{\xi}$ .

Due to the complexity of (19), we have to resort to numerical analysis to further understand the character of  $V[P]$ . The results of this exercise are discussed in Section 4.

### 3.7. Variance of net stock levels

From (6), we obtain the following expression:

$$\sum_{i=1}^{T_p} P_{t-i} + PIR_t = NS_{t+T_p+1} - NS_t + \sum_{i=1}^{T_p+1} D_{t+i} - FPIR_t - P_t. \tag{20}$$

Substituting (20) into (9) and rearranging it reveals that:

$$\begin{aligned} NS_{t+T_p+1} - TNS &= FPIR_t - \sum_{i=1}^{T_p+1} D_{t+i} - \left( E[FPIR_t] - E \left[ \sum_{i=1}^{T_p+1} D_{t+i} \right] \right) \\ &= FPIR_t - E[FPIR_t] - \left( \sum_{i=1}^{T_p+1} D_{t+i} - E \left[ \sum_{i=1}^{T_p+1} D_{t+i} \right] \right), \end{aligned} \quad (21)$$

indicating that the variance of the net stock level over an infinite time horizon,  $E[(NS_{t+T_p+1} - TNS)^2]$ , is equal to the expected value of the square of the right-hand side of (21):

$$\begin{aligned} V[NS] &= E[(NS_{t+T_p+1} - TNS)^2] \\ &= E[(FPIR_t - E[FPIR_t])^2] + E \left[ \left( \sum_{i=1}^{T_p+1} D_{t+i} - E \left[ \sum_{i=1}^{T_p+1} D_{t+i} \right] \right)^2 \right] \\ &\quad - 2E \left[ (FPIR_t - E[FPIR_t]) \left( \sum_{i=1}^{T_p+1} D_{t+i} - E \left[ \sum_{i=1}^{T_p+1} D_{t+i} \right] \right) \right]. \end{aligned}$$

The above expression shows that the net stock variance consists of: 1) the variance of the forecast error of  $FPIR_t$  over  $(T_p - T_r)$  time periods, 2) the variance of the forecast errors of the demand over  $(T_p + 1)$  time periods and 3) the covariance between those two forecast errors. If  $T_r \geq T_p$ ,  $FPIR_t$  is null and the net stock variance is identical to the variance of the forecast errors of the demand over  $T_p + 1$  time periods:

$$\begin{aligned} V[NS] &= E \left[ \left( \sum_{i=1}^{T_p+1} D_{t+i} - E \left[ \sum_{i=1}^{T_p+1} D_{t+i} \right] \right)^2 \right] \\ &= \frac{(T_p + 1)(1 - \phi_d^2) + \phi_d(1 - \phi_d^{T_p+1})(\phi_d^{T_p+2} - \phi_d - 2)}{(1 - \phi_d)^2(1 - \phi_d^2)} \sigma_d^2. \end{aligned} \quad (22)$$

This expression is identical to the variance of the net stock levels in a traditional forward supply chain facing an AR(1) demand with an OUT policy and MMSE forecasting scheme (Hosoda and Disney, 2006). This result is surprising because the manufacturer not only faces both uncertainty in demand but also uncertainty in serviceable returns. Equation (22) reveals the following properties:

**Property 4.** When  $T_r \geq T_p$  and  $\phi_d > 0$ ,  $V[NS]$  is increasing in  $T_p$ .

**Property 5.** When  $T_r \geq T_p$ ,  $V[NS]$  is independent of  $\{T_r, \phi_r, \theta_r, \sigma_r, \mu_r, \bar{\xi}, V[\xi]\}$ .

Property 4 suggests that a shorter manufacturing lead time,  $T_p$ , yields smaller inventory costs. This is intuitive and agrees with existing knowledge about traditional forward supply chains. Interestingly, as shown by Property 5, the remanufacturing lead time,  $T_r$ , does not affect  $V[NS]$  when  $T_r \geq T_p$ . Furthermore, when  $T_r \geq T_p$ , higher mean returns,  $\mu_r$ , do not affect the value of  $V[NS]$  either. Therefore, if  $T_r \geq T_p$  holds, the remanufacturer need not urgently process the returns, and higher collection rates do not increase the manufacturer's inventory variance.

When  $T_r < T_p$ , the net stock variance expression is rather complex. This complexity originates from the error terms in  $FPIR_t$ , which are correlated with both the demand and the return processes. To avoid clutter here, the expression for  $V[NS]$  when  $T_r < T_p$  is shown in Appendix 5 from which the following property can be obtained.

**Property 6.** *When  $T_r < T_p$ ,  $V[NS]$  is increasing in  $\mu_r$ .*

As shown in Appendix 5,  $V[NS]$  is increasing in  $\mu_r$ . Further characterisations of  $V[NS]$  in the case of  $T_r \leq T_p$  are explored in the next section using a numerical analysis.

#### 4. Numerical analysis of the lead-time paradox when $T_r \leq T_p$

Our interest herein is whether the lead-time paradox actually emerges when  $T_r \leq T_p$ . The error terms of the demand and the returns,  $\{\varepsilon_{d,t}, \varepsilon_{r,t}\}$ , are assumed to follow a normal distribution. For the yield rate,  $\xi_t$ , a triangular distribution with three parameters  $a$ ,  $b$  and  $c$  ( $0 \leq a \leq c \leq b \leq 1$ ) is assumed. Here,  $a$  and  $b$  are the range of support and  $c$  is the mode of the triangular distribution. The probability density function of the triage yield is given by:

$$f(\xi_t) = \begin{cases} 0, & \xi_t < a \\ \frac{2(\xi_t - a)}{(b-a)(c-a)}, & a \leq \xi_t < c \\ \frac{2}{b-a}, & \xi_t = c \\ \frac{2(b - \xi_t)}{(b-a)(b-c)}, & c < \xi_t \leq b \\ 0, & b < \xi_t. \end{cases}$$

The triage yield,  $\xi_t$ , has a mean of  $\bar{\xi} = (a + b + c)/3$  and a variance of  $V[\xi] = (a^2 + b^2 + c^2 - ab - ac - bc)/18$ . The following indicator is used to quantify the bullwhip effect:

$$BW = V[P]/V[D].$$

In addition, the standard deviation of the production order (as the capacity cost indicator),  $\sqrt{V[P]}$ , and that of the net stock levels (as the inventory cost indicator),  $\sqrt{V[NS]}$ , are used to measure the performance of our CLSC.

In our numerical analysis, unless otherwise stated, the following values are used:  $a = 0.5$ ,  $b = 0.99$ ,  $c = 0.8$ ,  $\mu_d = 100$ ,  $\mu_r = 50$ ,  $\sigma_d = \{1, 10\}$ ,  $\sigma_r = \{1, 10\}$ ,  $\phi_d = \{0.3, 0.7\}$ ,  $\phi_r = \{0.3, 0.7\}$ ,  $\theta_r = \{-0.9, 0, 0.9\}$  and  $T_p = 6$ . If we find that the value of  $BW$ ,  $\sqrt{V[P]}$  or  $\sqrt{V[NS]}$  is decreasing in  $T_r$ , we can conclude that the lead-time paradox occurs in the bullwhip ratios, the capacity cost or the inventory cost, respectively.

Figures 3–5 illustrate the relationships between those three indicators and  $T_r$ . In many settings, the cost indicators decrease in  $T_r$ , suggesting that longer remanufacturing lead times reduce those costs<sup>4</sup>. This is the evidence of the existence of the lead-time paradox. The results indicate that the lead-time paradox tends to be present when  $\sigma_r \gg \sigma_d$  or  $\theta_r < 0$ .

Figure 3 shows that the auto-correlation of the return process,  $\phi_r$ , has a greater impact on the lead-time paradox in  $BW$  than the auto-correlation of the demand process,  $\phi_d$ . In contrast, Figs. 4–5 suggest that  $\theta_r$  has a significant impact on  $\sqrt{V[P]}$  and  $\sqrt{V[NS]}$  when  $\sigma_d > \sigma_r$ . This particular result was investigated further, and the results are shown in Figs. 6–9.

Figures 6–7 show the impact of  $T_r$  on  $\sqrt{V[P]}$  when  $\phi_d = \phi_r = 0.3$  (Fig. 6) and  $\phi_d = \phi_r = 0.7$  (Fig. 7). Note that if we set  $\phi_d = \phi_r = 0$ ,  $\sqrt{V[P]}$  becomes independent of  $T_r$  (see (19)). Figure 6 reveals that when  $\theta_r$  is smaller and  $\sigma_r$  is bigger, the lead-time paradox is likely to be observed. Figure 7 shows that the lead-time paradox can appear under larger values of  $\theta_r$ . Figures 6–7 illustrate that when  $\theta_r = 0$ , the lead-time paradox in the capacity cost always emerges, irrespective of the values of  $\phi_d$ ,  $\phi_r$  and  $\sigma_r$ .

Figures 8–9 show the impact of  $T_r$  on  $\sqrt{V[NS]}$  when  $\phi_d = \phi_r = 0$  (Fig. 8) and  $\phi_d = \phi_r = 0.7$  (Fig. 9). Figure 8 suggests that when  $\theta_r = 0$ , the lead-time paradox always exists, irrespective of the value of  $\sigma_r$ . This finding supports the results of van der Laan et al. (1999) and Inderfurth and van der Laan (2001). Those studies used i.i.d. processes for the demand and the return, and no cross-correlation was assumed (i.e.  $\phi_d = \phi_r = \theta_r = 0$ ). In the case of the positive cross-correlation in Fig. 8, the lead-time paradox disappears, as  $\theta_r$  increases when  $\sigma_r$  is small. This finding coincides with the findings

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<sup>4</sup>Some lines in Fig. 3 are not easy to see for the case of  $\sigma_d = 10$  and  $\sigma_r = 1$ , so actual numbers are shown in Table 2 in Appendix 6.

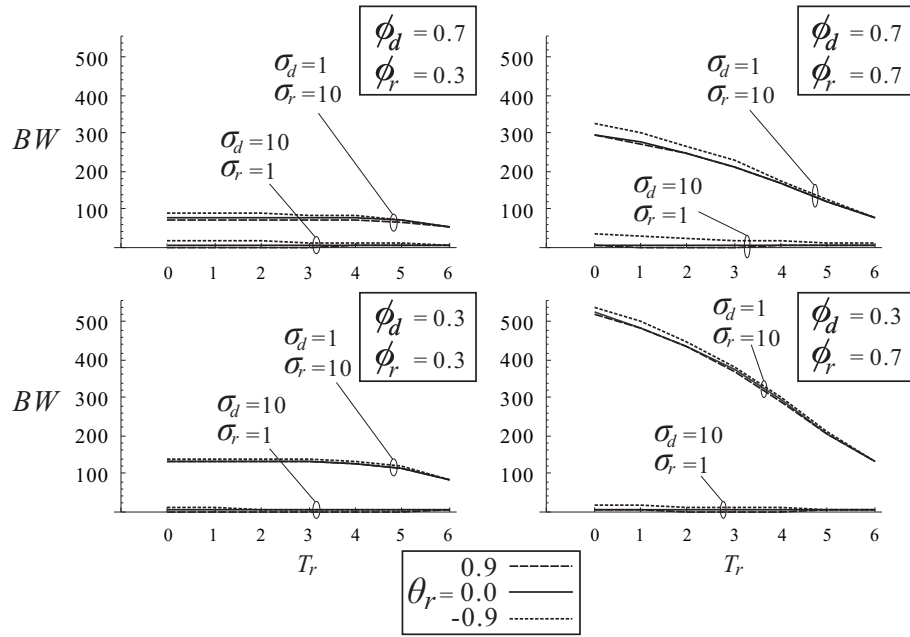


Figure 3: Lead-time paradox in bullwhip when  $T_r \leq T_p = 6$

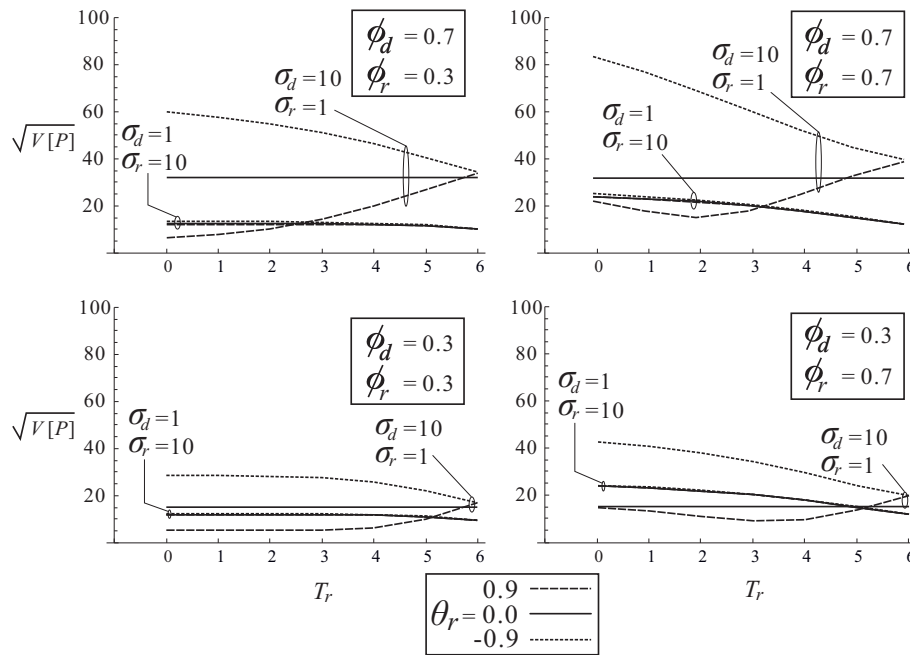


Figure 4: Lead-time paradox in capacity cost when  $T_r \leq T_p = 6$

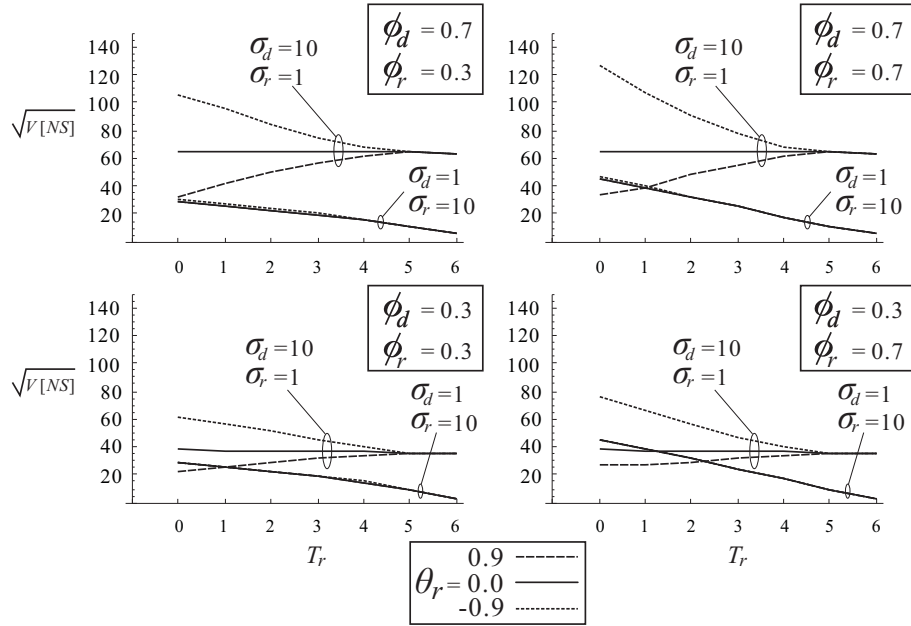


Figure 5: Lead-time paradox in inventory cost when  $T_r \leq T_p = 6$

of Zhou and Disney (2006). Their study assumed that the demand and the return are i.i.d. processes (i.e.  $\phi_d = \phi_r = 0$ ) and that they are positively correlated with each other (i.e.  $\theta_r > 0$ ). They concluded that shorter  $T_r$  can reduce the net stock variance.

Figure 9 shows the results of the positively auto-correlated case ( $\phi_d = \phi_r = 0.7$ ). In this case, the lead-time paradox is present if,  $\sigma_r > \sigma_d$ . The  $\theta_r = 0.9$  plot in Fig. 9 shows that when  $\sigma_r = \sigma_d$ , between  $T_r = 0$  and 1 there is a lead-time paradox and between  $T_r = 1$  and 5 there is no paradox but that it re-emerges between  $T_r = 5$  and 6. This is a rather complex set of behaviours. It is also reasonable to conclude that when the demand and the return are positively auto- and cross-correlated, a greater value of  $\sigma_r$  is necessary to observe the inventory lead-time paradox than to observe the lead-time paradox in orders.

Considering the results shown in this section, we may conclude that when  $T_r \leq T_p$ , shortening  $T_r$  may not be a good course of action when  $\theta_r \leq 0$  or  $\sigma_r \gg \sigma_d$ , as in such settings the lead-time paradox is evident. In addition, the cross-correlation assumption has a significant impact on the lead-time paradox. Table 1 shows a numerical example of the lead-time paradox when

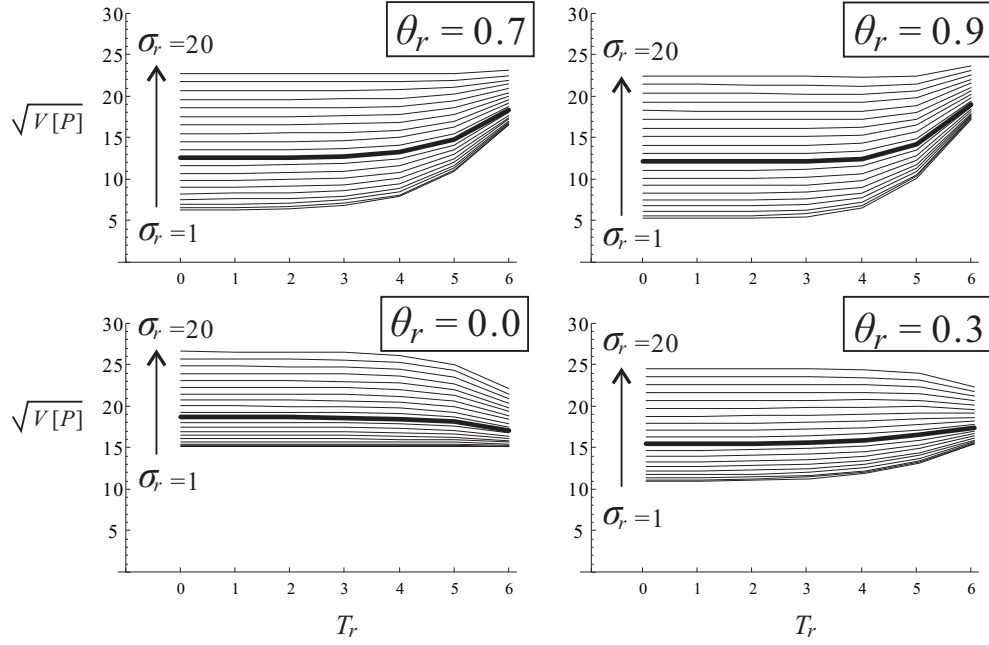


Figure 6: Impact of  $\sigma_r$  on the lead-time paradox in capacity cost when  $\sigma_d = 10$ ,  $\phi_d = \phi_r = 0.3$  and  $T_r \leq T_p = 6$  (the bold line represents the case of  $\sigma_r = \sigma_d = 10$ )

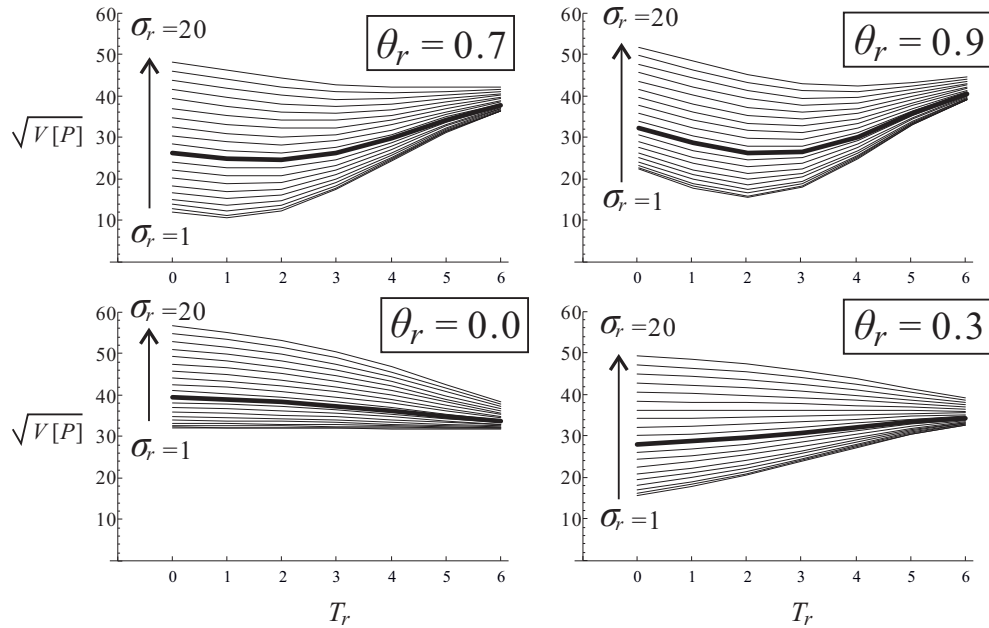


Figure 7: Impact of  $\sigma_r$  on the lead-time paradox in capacity cost when  $\sigma_d = 10$ ,  $\phi_d = \phi_r = 0.7$  and  $T_r \leq T_p = 6$  (the bold line represents the case of  $\sigma_r = \sigma_d = 10$ )



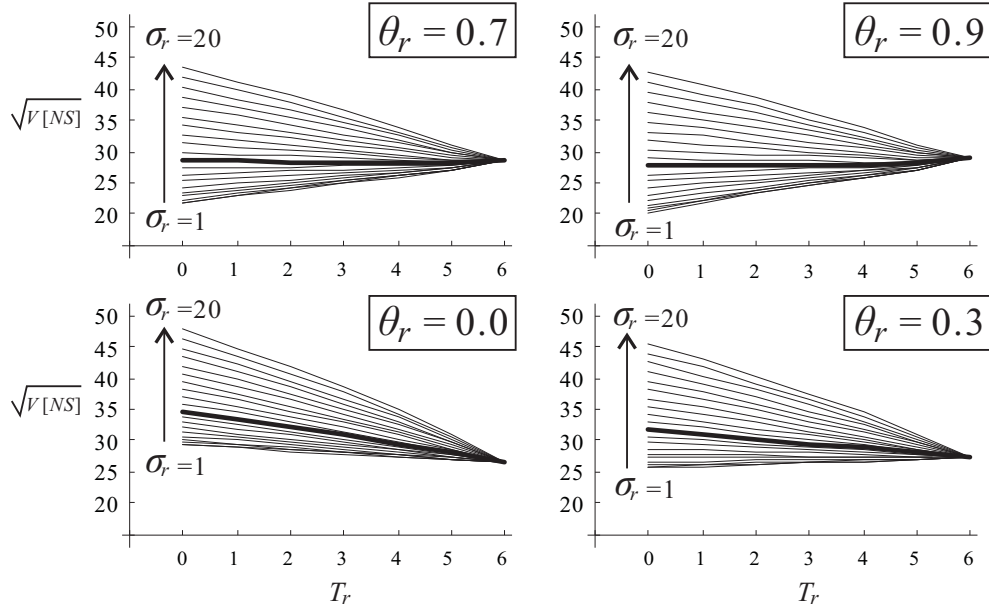


Figure 8: Impact of  $\sigma_r$  on the lead-time paradox in inventory cost when  $\sigma_d = 10$ ,  $\phi_d = \phi_r = 0.0$  and  $T_r \leq T_p = 6$  (the bold line represents the case of  $\sigma_r = \sigma_d = 10$ )

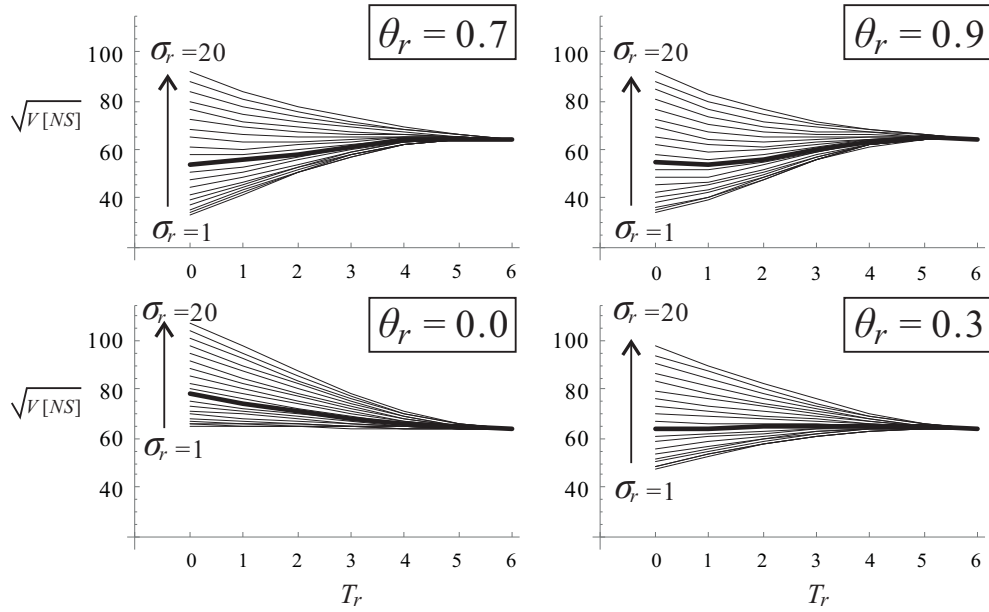


Figure 9: Impact of  $\sigma_r$  on the lead-time paradox in inventory cost when  $\sigma_d = 10$ ,  $\phi_d = \phi_r = 0.7$  and  $T_r \leq T_p = 6$  (the bold line represents the case of  $\sigma_r = \sigma_d = 10$ )

Table 1: The lead-time paradox when  $\phi_d = \phi_r = 0.7$ ,  $\theta_r = 0.3$ ,  $\sigma_d = 10$ ,  $\sigma_r = 20$  and  $T_r \leq T_p$

Case 1	Base case	Lengthening $T_r$			Shortening $T_p$		
$T_p$	3	3	3	3	2	1	0
$T_r$	0	1	2	3	0	0	0
$BW$	9.81	8.77	7.61	6.61	8.22	6.20	<b>4.08</b>
$\sqrt{V[P]}$	43.86	41.47	38.64	35.99	40.15	34.86	<b>28.29</b>
$\sqrt{V[NS]}$	57.40	48.73	42.16	38.86	41.82	25.60	<b>10.00</b>
Case 2	Base case	Lengthening $T_r$			Shortening $T_p$		
$T_p$	6	6	6	6	5	4	3
$T_r$	3	4	5	6	3	3	3
$BW$	10.65	9.75	8.72	7.79	9.52	8.10	<b>6.61</b>
$\sqrt{V[P]}$	45.70	43.72	41.34	39.08	43.20	39.85	<b>35.99</b>
$\sqrt{V[NS]}$	75.98	70.15	65.94	63.87	63.17	50.47	<b>38.86</b>

Note: Minimum value in each case is in bold.

$T_p = \{3, 6\}$ ,  $T_r = \{0, 3\}$  and  $\sigma_r \gg \sigma_d$ . To improve the performance of the CLSC, we consider two courses of action: 1) lengthening  $T_r$  while holding  $T_p$  constant or 2) shortening  $T_p$  while holding  $T_r$  constant. Table 1 suggests that both alternatives are attractive but that shortening  $T_p$  down to  $T_p = T_r = 0$  (Case 1) or  $T_p = T_r = 3$  (Case 2) can achieve the most economic performance. Once the manufacturer sets the lead times to identical values, then Properties 1–5 hold. Of the properties, Properties 1 and 4 suggest that shorter manufacturing lead time ( $T_p$ ) can yield lower bullwhip and capacity and inventory costs. Furthermore the impact of  $\theta_r$  on  $BW$  (see (17)) becomes minor, and its impact on  $\sqrt{V[NS]}$  disappears (Property 5).

## 5. Conclusions, managerial insights, and future research directions

Using auto- and cross-correlated demand and return processes, we investigated the dynamics of a CLSC with arbitrary lead times and a proportional random yield in the triage of returns. First, we derived the OUT policy with MMSE forecasting for the CLSC. This policy yields the minimum inventory cost for the manufacturer. It is assumed that the required information to enable this minimum cost policy is provided by the remanufacturer. We also highlighted some useful characteristics of auto- and cross-correlated demand and return processes via a detailed analysis of the processes. The dynamics of the CLSC were analysed both analytically and numerically. It

was shown that when  $T_r \geq T_p$ , the dynamics of the CLSC were similar to those observed in traditional forward supply chains. Furthermore, the inventory cost of the manufacturer is independent of the remanufacturing lead time. When  $T_r < T_p$ , the complex analytical results were supported by a numerical investigation. It was shown that when  $T_r < T_p$ , the lead-time paradox could emerge. This was the case not only in the inventory costs, but also in the bullwhip ratio and the capacity cost, especially when the returns are highly variable or non-positively correlated. This finding supports the results of van der Laan et al. (1999) and Inderfurth and van der Laan (2001). Our unifying theory also explains why the lead-time paradox is not observed in Zhou and Disney (2006). Finally, we recommend that the two lead times are first set equal by shortening  $T_p$  and then shortening them together when  $T_p = T_r$ . This helps to establish a more sustainable operation without sacrificing economic performance as improvements are made. It also helps the manufacturer to avoid the detrimental effects associated with the higher mean returns ( $\mu_r$ ) increasing its inventory cost when  $T_r < T_p$ . Once the relationship of  $T_r \geq T_p$  is established, such negative effects simply vanish. Ultimately, our managerial recommendation for manufacturers in CLSCs are:

**Rule 1** When the remanufacturing lead time is equal to or longer than the manufacturing lead time, shortening the manufacturing lead time reduces your capacity and inventory costs. Also in this setting, higher returns do not increase inventory costs. Shortening the remanufacturing lead time does not contribute to lower inventory costs but could generate some other benefits, such as lower capacity cost and in-transit inventory.

**Rule 2** When the remanufacturing lead time is less than the manufacturing lead time, you should understand that: a) the lead-time paradox can emerge, and b) higher mean returns always increase your inventory cost. Point a) suggests that shortening the remanufacturing lead time may not have desirable consequences. Point b) highlights the conflicting incentives between company performance and societal needs. To avoid these consequences, first shorten the manufacturing lead time until both lead times are equal. Then your incentives are aligned and Rule 1 applies.

In terms of potential future research directions, our demand and return model can be generalised further. One direction might be to use  $\theta_r(D_{t-\tau} - \mu_d)$ ,

where  $\tau = 1, 2, 3, \dots$ , instead of  $\theta_r(D_{t-1} - \mu_d)$  in (3). This affects the degree of the correlation between  $D_{t-\tau}$  and  $R_t$ . Another direction might be to consider adding cross-correlation from the previous returns to the current demand, to yield a more complete VAR(1) model. This model might be appropriate when the quality and accessibility of recycling facilities positively affects a market with a growing concern about environmental issues. Indeed, a full VARMA( $p, q$ ) model could be used to model the demand and returns, perhaps using a matrix-based approach as in Ratanachote (2011). Finally, in terms of the random yield model, correlation between the returns,  $R_t$ , and the triage yield,  $\xi_t$ , might better reflect reality. A simulation approach may be needed to address this research direction.

### Acknowledgements

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### Appendix 1. The variances of and covariances between the demand and the returns

In this section, we identify the variances of  $D_t$  and  $R_t$ , the covariance between  $D_{t-n}$  and  $R_t$  ( $n = 1, 2, 3, \dots$ ) and the covariance between  $D_t$  and  $R_{t-n}$ . Generally, the variance of a random variable  $x$  is denoted by  $E[(x - E[x])^2]$ . Then, because  $E[D_t] = \mu_d$ , the variance of AR(1) process  $D_t$  is:

$$\begin{aligned} V[D] &= E[(D_t - E[D_t])^2] \\ &= E[(\phi_d(D_{t-1} - \mu_d) + \varepsilon_{d,t})^2] \\ &= \phi_d^2 V[D] + \sigma_d^2. \end{aligned} \tag{23}$$

Solving the equation above for  $V[D]$ , we obtain the demand variance:

$$V[D] = \frac{\sigma_d^2}{1 - \phi_d^2}.$$

Box et al. (2008) show that the autocovariance between  $D_t$  and  $D_{t-n}$  ( $n = 1, 2, 3, \dots$ ) is:

$$E[(D_t - \mu_d)(D_{t-n} - \mu_d)] = \phi_d^n V[D]. \tag{24}$$

Using  $E[R_t] = \mu_r$  and  $E[\varepsilon_{r,t}] = 0$ , we can describe the variance of the returns as follows:

$$\begin{aligned} V[R] &= E[(R_t - E[R_t])^2] \\ &= E[(\phi_r(R_{t-1} - \mu_r) + \theta_r(D_{t-1} - \mu_d) + \varepsilon_{r,t})^2] \\ &= \phi_r^2 V[R] + \theta_r^2 V[D] + \sigma_r^2 + 2\phi_r\theta_r COV_0, \end{aligned} \quad (25)$$

where  $COV_0$  is the covariance between  $R_t$  and  $D_t$ . From the definition of the covariance, we have:

$$\begin{aligned} COV_0 &= E[(R_t - \mu_r)(D_t - \mu_d)] \\ &= E[(\phi_r(R_{t-1} - \mu_r) + \theta_r(D_{t-1} - \mu_d) + \varepsilon_{r,t})(\phi_d(D_{t-1} - \mu_d) + \varepsilon_{d,t})] \\ &= \phi_d\phi_r COV_0 + \theta_r\phi_d V[D]. \end{aligned} \quad (26)$$

To obtain (26), we have used the relation  $E[(R_{t-1} - \mu_r)(D_{t-1} - \mu_d)] = E[(R_t - \mu_r)(D_t - \mu_d)]$ . Rearranging (26) yields  $COV_0$ :

$$COV_0 = \frac{\phi_d\theta_r}{1 - \phi_d\phi_r} V[D]. \quad (27)$$

Finally, we obtain the following expression for the variance of the returns:

$$V[R] = \frac{\theta_r^2(1 + \phi_d\phi_r)V[D] + (1 - \phi_d\phi_r)\sigma_r^2}{(1 - \phi_d\phi_r)(1 - \phi_r^2)}.$$

Using the knowledge of  $COV_0$ , the covariance between  $R_t$  and  $D_{t-1}$  (as this is the case where  $n = 1$ ),  $COV_1$ , can be written as:

$$\begin{aligned} COV_1 &= E[(R_t - \mu_r)(D_{t-1} - \mu_d)] \\ &= E[(\phi_r(R_{t-1} - \mu_r) + \theta_r(D_{t-1} - \mu_d) + \varepsilon_{r,t})(D_{t-1} - \mu_d)] \\ &= \phi_r COV_0 + \theta_r V[D] \\ &= \frac{\theta_r}{1 - \phi_d\phi_r} V[D]. \end{aligned}$$

Furthermore,  $COV_2$  is

$$\begin{aligned} COV_2 &= E[(R_t - \mu_r)(D_{t-2} - \mu_d)] \\ &= E[(\phi_r(\phi_r(R_{t-2} - \mu_r) + \theta_r(D_{t-2} - \mu_d) + \varepsilon_{r,t-1}) \\ &\quad + \theta_r(\phi_d(D_{t-2} - \mu_d) + \varepsilon_{d,t-1}) + \varepsilon_{r,t})(D_{t-2} - \mu_d)] \\ &= \phi_r^2 COV_0 + \theta_r(\phi_d + \phi_r)V[D]. \end{aligned}$$

Repeating these same steps, we can then use induction to find an expression for  $COV_n$  ( $n \geq 2$ )

$$\begin{aligned} COV_{n \geq 2} &= E[(R_t - \mu_r)(D_{t-n} - \mu_d)] \\ &= \begin{cases} \phi_r^n COV_0 + \theta_r \frac{\phi_d^n - \phi_r^n}{\phi_d - \phi_r} V[D], & \phi_d \neq \phi_r \\ \phi_r^n COV_0 + n\theta_r \phi_d^{n-1} V[D], & \phi_d = \phi_r \neq 0 \\ \theta_r \phi_r^{n-1} V[D], & \phi_d = 0 \wedge \phi_r \neq 0 \\ \theta_r \phi_d^{n-1} V[D], & \phi_d \neq 0 \wedge \phi_r = 0 \\ 0, & \phi_d = \phi_r = 0. \end{cases} \end{aligned}$$

The knowledge of  $COV_n$  enables us to obtain an expression of the correlation between  $R_t$  and  $D_{t-n}$ ,  $\rho_n = COV_n / (\sqrt{V[D]}\sqrt{V[R]})$ .

Following similar steps to those shown above, we can also find that the covariance between  $D_t$  and  $R_{t-n}$  ( $n = 1, 2, 3, \dots$ ) is:

$$E[(D_t - \mu_d)(R_{t-n} - \mu_r)] = \phi_d^n COV_0. \quad (28)$$

## Appendix 2. The variance of the serviceable returns

The variance of the returns subject to the random yield when the return process follows a VAR(1) process is derived as follows:

$$\begin{aligned} V[\Xi[R]] &= V[\xi_t R_t] = E[(\xi_t R_t - E[\xi_t R_t])^2] \\ &= E[(\xi_t - \bar{\xi})\mu_r + \xi_t \phi_r (R_{t-1} - \mu_r) + \xi_t \theta_r (D_{t-1} - \mu_d) + \xi_t \varepsilon_{r,t})^2] \\ &= V[\xi] \mu_r^2 + (V[\xi] + \bar{\xi}^2)(\phi_r^2 V[R] + \theta_r^2 V[D] + \sigma_r^2 + 2\phi_r \theta_r COV_0) \\ &= V[\xi] \mu_r^2 + (V[\xi] + \bar{\xi}^2) V[R]. \end{aligned}$$

To obtain this result, we used (25) and (27).

## Appendix 3. Normality tests

We conducted normality tests on the simulated values of  $NS_t$  and  $P_t$  when  $\varepsilon_{d,t}$  and  $\varepsilon_{r,t}$  were normally distributed but  $\xi_t$  was drawn from a triangular distribution. The data set was generated by a 20,000 time period numerical simulation. For the simulation, the following parameters were used:  $\mu_d = 100$ ,  $\mu_r = 50$ ,  $\phi_d = \phi_r = 0.7$ ,  $\theta_r = 0.3$ ,  $T_p = 1$ ,  $T_r = 0$  and  $TNS = 10$ . The error terms,  $\varepsilon_{d,t}$  and  $\varepsilon_{r,t}$ , follow  $N(0, 10^2)$  and  $N(0, 20^2)$ , respectively. The

values of  $\{a, b, c\}$  for the triangular distribution were: Case 1  $\{0.3, 0.7, 0.5\}$  and Case 2  $\{0.5, 0.99, 0.8\}$ . Figure 10 shows the quantile-quantile (Q-Q) plots of the simulation results with the  $p$ -values of a Kolmogorov-Smirnov test for each case. The null hypothesis, which is the sample data set is drawn from a normal distribution, was not rejected in any of the cases.

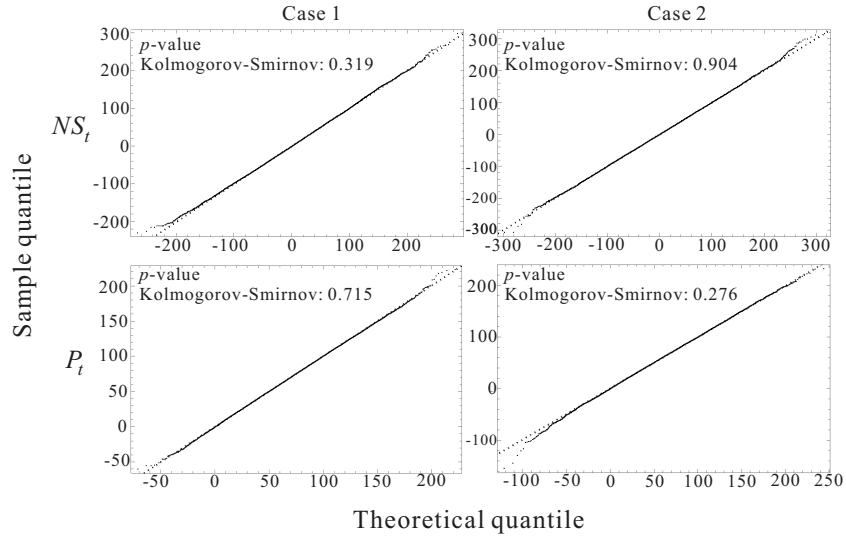


Figure 10: Q-Q plots of  $NS_t$ (upper) and  $P_t$ (lower) when  $\{a, b, c\} = \{0.3, 0.7, 0.5\}$  (left) and  $\{0.5, 0.99, 0.8\}$  (right).

#### Appendix 4. The variance of the production orders when $T_r \geq T_p$

The process to obtain  $V[P]$  for the case of  $T_r \geq T_p$  is shown below.

$$\begin{aligned}
V[P] &= E[(P_t - E[P_t])^2] \\
&= E \left[ ((D_t - \mu_d) - (\Xi[R_{t-(T_r-T_p)}] - \bar{\xi}\mu_r) \right. \\
&\quad \left. + \Lambda_d(\phi_d - 1)(D_{t-1} - \mu_d) + \Lambda_d\varepsilon_{d,t})^2 \right] \\
&= V[D] + V[\Xi[R]] + \Lambda_d^2(\phi_d - 1)^2 V[D] + \Lambda_d^2\sigma_d^2 - 2\bar{\xi}\phi_d^{T_r-T_p} COV_0 \\
&\quad + 2\Lambda_d(\phi_d - 1)\phi_d V[D] + 2\Lambda_d\sigma_d^2 - 2\bar{\xi}\Lambda_d(\phi_d - 1)\phi_d^{T_r-T_p-1} COV_0, \\
&= V[D] + V[\Xi[R]] + \Lambda_d^2(\phi_d^2 - 2\phi_d + 1)V[D] + \Lambda_d^2\sigma_d^2 - 2\bar{\xi}\phi_d^{T_r-T_p} COV_0 \\
&\quad + 2\phi_d^2\Lambda_d V[D] - 2\phi_d\Lambda_d V[D] + 2\Lambda_d\sigma_d^2 - 2\bar{\xi}\Lambda_d(\phi_d - 1)\phi_d^{T_r-T_p-1} COV_0, \\
&= V[D] + V[\Xi[R]] + \underbrace{\Lambda_d^2(\phi_d^2 V[D] + \sigma_d^2)}_{=V[D]} - 2\phi_d\Lambda_d^2 V[D] + \Lambda_d^2 V[D] \\
&\quad - 2\bar{\xi}\phi_d^{T_r-T_p} COV_0 + 2\Lambda_d \underbrace{(\phi_d^2 V[D] + \sigma_d^2)}_{=V[D]} \\
&\quad - 2\phi_d\Lambda_d V[D] - 2\bar{\xi}\Lambda_d(\phi_d - 1)\phi_d^{T_r-T_p-1} COV_0, \\
&= V[D] + V[\Xi[R]] + 2\Lambda_d^2 V[D] - 2\phi_d\Lambda_d^2 V[D] - 2\bar{\xi}\phi_d^{T_r-T_p} COV_0 \\
&\quad + 2\Lambda_d V[D] - 2\phi_d\Lambda_d V[D] - 2\bar{\xi}\Lambda_d(\phi_d - 1)\phi_d^{T_r-T_p-1} COV_0, \\
&= V[D] + V[\Xi[R]] + 2\Lambda_d^2 V[D](1 - \phi_d) + 2\Lambda_d V[D](1 - \phi_d) \\
&\quad - 2\bar{\xi}\phi_d^{T_r-T_p-1} \underbrace{(\phi_d + \Lambda_d(\phi_d - 1))}_{=\phi_d^{T_p+2}} COV_0, \\
&= V[D] + V[\Xi[R]] + 2\Lambda_d \underbrace{(1 - \phi_d)(\Lambda_d + 1)}_{=1-\phi_d^{T_p+2}} V[D] - 2\bar{\xi}\phi_d^{T_r+1} COV_0 \\
&= V[D] + V[\Xi[R]] + 2\Lambda_d(1 - \phi_d^{T_p+2}) V[D] - 2\bar{\xi}\phi_d^{T_r+1} COV_0,
\end{aligned}$$

where  $\Lambda_d = \phi_d(1 - \phi_d^{T_p+1})/(1 - \phi_d)$ . To simplify this expression, (23), (24) and (27) were used.



### Appendix 5. The variance of the net stock levels when $T_r < T_p$

The variance of the net stock levels when  $T_r < T_p$  can be described as follows:

$$\begin{aligned}
 V[NS] = & E[(FPIR_t - E[FPIR_t])^2] + E\left[\left(\sum_{i=1}^{T_p+1} D_{t+i} - E\left[\sum_{i=1}^{T_p+1} D_{t+i}\right]\right)^2\right] \\
 & - 2E\left[(FPIR_t - E[FPIR_t])\left(\sum_{i=1}^{T_p+1} D_{t+i} - E\left[\sum_{i=1}^{T_p+1} D_{t+i}\right]\right)\right] \\
 = & E\left[\underbrace{\left(\sum_{i=1}^{T_p-T_r} \xi_{t+i} R_{t+i} - E\left[\sum_{i=1}^{T_p-T_r} \xi_{t+i} R_{t+i}\right]\right)^2}_A\right] \\
 & + E\left[\underbrace{\left(\sum_{i=1}^{T_p+1} D_{t+i} - E\left[\sum_{i=1}^{T_p+1} D_{t+i}\right]\right)^2}_B\right] \\
 & - 2E\left[\underbrace{\left(\sum_{i=1}^{T_p-T_r} \xi_{t+i} R_{t+i} - E\left[\sum_{i=1}^{T_p-T_r} \xi_{t+i} R_{t+i}\right]\right)}_C \times \left(\sum_{i=1}^{T_p+1} D_{t+i} - E\left[\sum_{i=1}^{T_p+1} D_{t+i}\right]\right)\right].
 \end{aligned}$$

The expressions of  $A$ ,  $B$  and  $C$  are dependent upon the values of  $\phi_d$  and  $\phi_r$ :

$$\begin{aligned}
 A = & E\left[\left(\sum_{i=1}^{T_p-T_r} \xi_{t+i} R_{t+i} - E\left[\sum_{i=1}^{T_p-T_r} \xi_{t+i} R_{t+i}\right]\right)^2\right] \\
 = & E\left[\left(\sum_{i=1}^{T_p-T_r} \xi_{t+i} R_{t+i} - \bar{\xi} E[R_{t+i}]\right)^2\right] \\
 = & E\left[\left(\sum_{i=1}^{T_p-T_r} \xi_{t+i} (E[R_{t+i}] + R_{t+i} - E[R_{t+i}]) - \bar{\xi} E[R_{t+i}]\right)^2\right] \\
 = & E\left[\left(\sum_{i=1}^{T_p-T_r} (\xi_{t+i} - \bar{\xi}) E[R_{t+i}] + \xi_{t+i} (R_{t+i} - E[R_{t+i}])\right)^2\right] \\
 = & \underbrace{V[\xi] \sum_{i=1}^{T_p-T_r} E[R_{t+i}]^2}_{A_1} + \underbrace{\sum_{i=1}^{T_p-T_r} E[\xi_{t+i}^2] E[(R_{t+i} - E[R_{t+i}])^2]}_{A_2} \\
 & + E\left[\underbrace{\left(\sum_{i=1}^{T_p-T_r} \sum_{j \in \{1, \dots, T_p-T_r\} \setminus i} \xi_{t+i} (R_{t+i} - E[R_{t+i}]) \xi_{t+j} (R_{t+j} - E[R_{t+j}])\right)^2}_{A_3}\right],
 \end{aligned}$$

where

$$\begin{aligned}
A_1 &= \begin{cases} V[\xi] \sum_{i=1}^{T_p-T_r} \left( \mu_r^2 + \phi_r^{2i} V[R] + \theta_r^2 \left( \frac{\phi_d^i - \phi_r^i}{\phi_d - \phi_r} \right)^2 V[D] \right. \\ \left. + \frac{2\phi_d \phi_r^i \theta_r^2 (\phi_d^i - \phi_r^i)}{(\phi_d - \phi_r)(1 - \phi_d \phi_r)} V[D] \right), & \phi_d \neq \phi_r \\ V[\xi] (T_p - T_r) (\mu_r^2 + \theta_r^2 \sigma_d^2), & \phi_d = \phi_r = 0 \\ V[\xi] \sum_{i=1}^{T_p-T_r} \left( \mu_r^2 + \phi_r^{2i} V[R] + \theta_r^2 (i \phi_d^{i-1})^2 V[D] \right. \\ \left. + \frac{2i \theta_r^2 \phi_d^{2i}}{1 - \phi_d^2} V[D] \right), & \phi_d = \phi_r \neq 0, \end{cases} \\
A_2 &= \begin{cases} (V[\xi] + \bar{\xi}^2) \\ \times \sum_{i=1}^{T_p-T_r} \left( \sum_{j=0}^{i-1} \phi_r^{2j} \sigma_r^2 + \theta_r^2 \sum_{j=0}^{i-1} \left( \frac{\phi_d^j - \phi_r^j}{\phi_d - \phi_r} \right)^2 \sigma_d^2 \right), & \phi_d \neq \phi_r \wedge \phi_d \phi_r \neq 0 \\ (V[\xi] + \bar{\xi}^2) \sum_{i=1}^{T_p-T_r} \left( \sum_{j=0}^{i-1} \phi_r^{2j} \sigma_r^2 + \theta_r^2 \sum_{j=0}^{i-2} \phi_r^{2j} \sigma_d^2 \right), & \phi_d = 0 \wedge \phi_r \neq 0 \\ (V[\xi] + \bar{\xi}^2) \sum_{i=1}^{T_p-T_r} \left( \sigma_r^2 + \theta_r^2 \sum_{j=0}^{i-2} \phi_d^{2j} \sigma_d^2 \right), & \phi_d \neq 0 \wedge \phi_r = 0 \\ (V[\xi] + \bar{\xi}^2) ((T_p - T_r) \sigma_r^2 + (T_p - T_r - 1) \theta_r^2 \sigma_d^2), & \phi_d = \phi_r = 0 \\ (V[\xi] + \bar{\xi}^2) \\ \times \sum_{i=1}^{T_p-T_r} \left( \sum_{j=0}^{i-1} \phi_r^{2j} \sigma_r^2 + \theta_r^2 \sum_{j=0}^{i-1} (j \phi_d^{j-1})^2 \sigma_d^2 \right), & \phi_d = \phi_r \neq 0, \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
A_3 &= \begin{cases} 2 \sum_{i=1}^{T_p-T_r} \left( \bar{\xi}^2 \sum_{j=1}^{T_p-T_r-i} \sum_{k=j+1}^{T_p-T_r+1-i} \phi_r^{j+k-2} \sigma_r^2 \right. \\ \left. + \theta_r^2 \bar{\xi}^2 \sum_{j=1}^{T_p-T_r-1-i} \sum_{k=j+1}^{T_p-T_r-i} \frac{(\phi_d^j - \phi_r^j)(\phi_d^k - \phi_r^k)}{(\phi_d - \phi_r)^2} \sigma_d^2 \right), & \phi_d \neq \phi_r \\ 0, & \phi_d = \phi_r = 0 \\ 2 \sum_{i=1}^{T_p-T_r} \left( \bar{\xi}^2 \sum_{j=1}^{T_p-T_r-i} \sum_{k=j+1}^{T_p-T_r+1-i} \phi_r^{j+k-2} \sigma_r^2 \right. \\ \left. + \theta_r^2 \bar{\xi}^2 \sum_{j=1}^{T_p-T_r-1-i} \sum_{k=j+1}^{T_p-T_r-i} j k \phi_d^{j+k-2} \sigma_d^2 \right), & \phi_d = \phi_r \neq 0. \end{cases}
\end{aligned}$$

$$\begin{aligned}
B &= E \left[ \left( \sum_{i=1}^{T_p+1} D_{t+i} - E \left[ \sum_{i=1}^{T_p+1} D_{t+i} \right] \right)^2 \right] \\
&= \frac{(T_p + 1)(1 - \phi_d^2) + \phi_d(1 - \phi_d^{T_p+1})(\phi_d^{T_p+2} - \phi_d - 2)}{(1 - \phi_d)^2(1 - \phi_d^2)} \sigma_d^2.
\end{aligned}$$

$$\begin{aligned}
C &= -2E \left[ \left( \sum_{i=1}^{T_p-T_r} \xi_{t+i} R_{t+i} - E \left[ \sum_{i=1}^{T_p-T_r} \xi_{t+i} R_{t+i} \right] \right) \right. \\
&\quad \times \left. \left( \sum_{i=1}^{T_p+1} D_{t+i} - E \left[ \sum_{i=1}^{T_p+1} D_{t+i} \right] \right) \right] \\
&= \begin{cases} -2\theta_r \bar{\xi} \sum_{i=1}^{T_p-T_r} \sum_{j=1}^{T_p-T_r-i} \frac{(\phi_d^i - \phi_r^i)(1 - \phi_d^{T_p+2-j})}{(\phi_d - \phi_r)(1 - \phi_d)} \sigma_d^2, & \phi_d \neq \phi_r \\ -2\theta_r \bar{\xi} (T_p - T_r - 1) \sigma_d^2, & \phi_d = \phi_r = 0 \\ -2\theta_r \bar{\xi} \sum_{i=1}^{T_p-T_r} \sum_{j=1}^{T_p-T_r-i} \frac{i \phi_d^{i-1} (1 - \phi_d^{T_p+2-j})}{1 - \phi_d} \sigma_d^2, & \phi_d = \phi_r \neq 0. \end{cases}
\end{aligned}$$

### Appendix 6. The lead-time paradox in bullwhip when $T_r \leq T_p = 6$ , $\sigma_d = 10$ and $\sigma_r = 1$

Figure 3 illustrates the presence of the lead-time paradox in the bullwhip measure. However, the case when  $\sigma_d = 10$  and  $\sigma_r = 1$  was hard to see in Fig. 3. Table 2 provides an alternative visualisation for clarity.

Table 2: The values of  $BW$  when  $\sigma_d = 10$  and  $\sigma_r = 1$ , used in Fig. 3

$T_r$	0	1	2	3	4	5	6
$\phi_d = 0.7, \phi_r = 0.7$							
$\theta = 0.9$	2.555	1.626	<b>1.216</b>	1.655	3.169	5.561	7.767
$\theta = 0.0$	5.193	5.191	5.188	5.185	5.180	5.175	<b>5.171</b>
$\theta = -0.9$	35.504	30.174	24.520	18.961	14.042	10.292	<b>8.077</b>
$\phi_d = 0.7, \phi_r = 0.3$							
$\theta = 0.9$	<b>0.216</b>	0.320	0.550	1.034	1.994	3.692	5.876
$\theta = 0.0$	5.171	5.171	5.171	5.171	5.171	5.170	<b>5.169</b>
$\theta = -0.9$	18.194	17.025	15.451	13.409	10.927	8.265	<b>6.077</b>
$\phi_d = 0.3, \phi_r = 0.7$							
$\theta = 0.9$	2.055	1.623	1.159	<b>0.798</b>	0.844	1.762	3.541
$\theta = 0.0$	2.138	2.135	2.130	2.123	2.115	2.106	<b>2.099</b>
$\theta = -0.9$	16.616	15.140	13.197	10.765	7.989	5.335	<b>3.541</b>
$\phi_d = 0.3, \phi_r = 0.3$							
$\theta = 0.9$	<b>0.254</b>	0.255	0.258	0.278	0.388	0.943	2.726
$\theta = 0.0$	2.099	2.099	2.099	2.099	2.099	2.097	<b>2.094</b>
$\theta = -0.9$	7.517	7.465	7.324	6.958	6.104	4.516	<b>2.726</b>

Note: Minimum value in each case is in bold.

## References

- Akçali, E., Çetinkaya, S., 2011. Quantitative models for inventory and production planning in closed-loop supply chains. *International Journal of Production Research* 49, 2373–2407.
- Ali, M. M., Babai, M. Z., Boylan, J. E., Syntetos, A. A., 2017. Supply chain forecasting when information is not shared. *European Journal of Operational Research*. *In Press*. DOI: 10.1016/j.ejor.2016.11.046.
- Alwan, L. C., Liu, J. J., Yao, D., 2003. Stochastic characterization of upstream demand process in a supply chain. *IIE Transactions* 35, 207–219.
- Boute, R. N., Dejonckheere, J., Disney, S. M., Van de Velde, W., 2013. Bullwhip in a multi-product production setting. *Tijdschrift voor Economie en Management—Review of Business and Economic Literature* 57, 354–381.
- Box, G. E. P., Jenkins, G. M., Reinsel, G. C., 2008. *Time Series Analysis: Forecasting and Control*, 4th Edition. John Wiley & Sons, NJ.
- Box, G. E. P., Tiao, G. C., 1977. A canonical analysis of multiple time series. *Biometrika* 64, 355–365.
- Cannella, S., Bruccoleri, M., Framinan, J. M., 2016. Closed-loop supply chains: What reverse logistics factors influence performance? *International Journal of Production Economics* 175, 35–49.
- Chaharsooghi, S. K., Sadeghi, A., 2008. On the bullwhip effect measure in supply chains with VAR(1) demand process. *International Journal of Industrial Engineering & Production Research* 19, 9–19.
- Chen, Y. F., Drezner, Z., Ryan, J. K., Simchi-Levi, D., 2000. Quantifying the bullwhip effect in a simple supply chain: The impact of forecasting, lead times, and information. *Management Science* 46 (3), 436–443.
- Coelho, T. M., Castro, R., Gobbo Jr., J. A., 2011. PET containers in Brazil: Opportunities and challenges of a logistics model for post-consumer waste recycling. *Resources, Conservation and Recycling* 55, 291–299.
- CPBR, 2012. Pet bottle recycling in Japan. Council for PET Bottle Recycling. <http://www.petbottle-rec.gr.jp/english/reports.html>. [accessed on Feb. 10, 2017]

- CPBR, 2015. Recycling Rate of PET Bottles. Council for PET Bottle Recycling. <http://www.petbottle-rec.gr.jp/english/actual2.html>. [accessed on Feb. 10, 2017]
- Dejonckheere, J., Disney, S. M., Lambrecht, M. R., Towill, D. R., 2003. Measuring and avoiding the bullwhip effect: A control theoretic approach. *European Journal of Operational Research* 147, 567–590.
- Disney, S. M., Hoshiko, L., Polley, L. and Weigel, C., 2013. Removing bullwhip from Lexmark's toner operations, *Production and Operations Management Society Annual Conference*, Denver, USA, May 3rd – 6th, 10 pages.
- Disney, S. M., Gaalman, G. and Hosoda, T., 2012. Review of stochastic cost functions for production and inventory control. Pre-prints of the 17th International Working Seminar of Production Economics, Innsbruck, Austria, February 20th – 24th, 1, 117–128.
- Disney, S. M., Towill, D. R., 2003. On the bullwhip and inventory variance produced by an ordering policy. *OMEGA, The International Journal of Management Science* 31, 157–167.
- Hedenstierna, C. P. T., and Disney, S. M., (2016), Inventory performance under staggered deliveries and auto-correlated demand, *European Journal of Operational Research*, 249, 3, 1082–1091.
- Henig, M., Gerchak, Y., 1990. The structure of periodic review policies in the presence of random yield. *Operations Research* 38, 634–643.
- Hosoda, T., Disney, S. M., 2006. On variance amplification in a three-echelon supply chain with minimum mean square error forecasting. *OMEGA, The International Journal of Management Science* 34 (4), 344–358.
- Hosoda, T., Disney, S. M., Gavirneni, S., 2015. The impact of information sharing, random yield, correlation, and lead times in closed loop supply chains. *European Journal of Operational Research* 246, 827–836.
- Hosoda, T., Naim, M. M., Disney, S. M., Potter, A., 2008. Is there a benefit to sharing market sales information? Linking theory and practice. *Computers & Industrial Engineering* 54, 315–326.

- Inderfurth, K., van der Laan, E., 2001. Leadtime effects and policy improvement for stochastic inventory control with remanufacturing. *International Journal of Production Economics* 71, 381–390.
- Kim, H., Ryan, J. K., 2003. The cost impact of using simple forecasting techniques in a supply chain. *Naval Research Logistics* 50, 388–411.
- Kurata, H., Liu, J. J., Yao, D., 2007. Analysis of supply chain's strategic resolutions to order variability - risk pooling and bundling - under vector autoregressive demand. *International Journal of Operations and Quantitative Management* 13, 173–198.
- Lee, H. L., Padmanabhan, V., Whang, S., 1997. The bullwhip effect in supply chains. *Sloan Management Review* 38 (3), 93–102.
- Lee, H. L., So, K. C., Tang, C. S., 2000. The value of information sharing in a two-level supply chain. *Management Science* 46 (5), 626–643.
- Potter, A. and Disney, S. M., 2010. Removing bullwhip from the Tesco supply chain. *Production and Operations Management Society Annual Conference*, May 7th – 10th, Vancouver, Canada. Paper No. 015-0397, 19 pages.
- PRE, 2016a. Plastics recycling resists low oil prices. *Plastic Recyclers Europe*. <http://www.plasticsrecyclers.eu/news/plastics-recycling-resists-low-oil-prices>. [accessed on Feb. 10, 2017]
- PRE, 2016b. 20 years later & the way forward. *Plastic Recyclers Europe*. <http://www.plasticsrecyclers.eu/sites/default/files/PRE%20Strategy%20Paper%202016%20v2.0.pdf>. [accessed on Feb. 10, 2017]
- Raghunathan, S., Tang, C. S., Yue, X., 2017. Analysis of the bullwhip effect in a multi-product setting with interdependent demands. *Operations Research*. *In Press*. DOI: <http://dx.doi.org/10.1287/opre.2016.1571>.
- Ratanachote, P., 2011. Distribution network dynamics with correlated demands. Ph.D. Thesis, Cardiff Business School, Cardiff University, UK.
- Sadeghi, A., 2015. Providing a measure for bullwhip effect in a two-product supply chain with exponential smoothing forecasts. *International Journal of Production Economics* 169, 44–54.

- Suntory Group, 2013. Suntory Group Corporate Social Responsibility Report 2013. [http://www.suntory.com/csr/report/pdf/2013/report\\_all.pdf](http://www.suntory.com/csr/report/pdf/2013/report_all.pdf). [accessed on Feb. 10, 2017]
- Tiao, G. C., Box, G. E. P., 1981. Modeling multiple times series with applications. *Journal of the American Statistical Association* 76, 802–816.
- Van der Laan, E., Salomon, M., Dekker, R., 1999. An investigation of lead-time effects in manufacturing/remanufacturing systems under simple push and pull control strategies. *European Journal of Operational Research* 115, 195–214.
- Vassian, H. J., 1955. Application of discrete variable servo theory to inventory control. *Journal of the Operations Research Society of America* 3 (3), 272–282.
- Wang, X., Disney, S. M., 2016. The bullwhip effect: Progress, trends and directions. *European Journal of Operational Research* 250, 691–701.
- Welle, F., 2011. Twenty years of PET bottle to bottle recycling - An overview. *Resources, Conservation and Recycling* 55, 865–875.
- Yano, C. A., Lee, H. L., 1995. Lot sizing with random yields: A review. *Operations Research* 43, 311–334.
- Zhang, X., 2004. The impact of forecasting methods on the bullwhip effect. *International Journal of Production Economics* 88, 15–27.
- Zhou, L., Disney, S. M., 2006. Bullwhip and inventory variance in a closed loop supply chain. *OR Spectrum* 28, 127–149.
- Zipkin, P. H., 2000. *Foundations of Inventory Management*. McGraw-Hill.